One dimensional Fluid Flow:

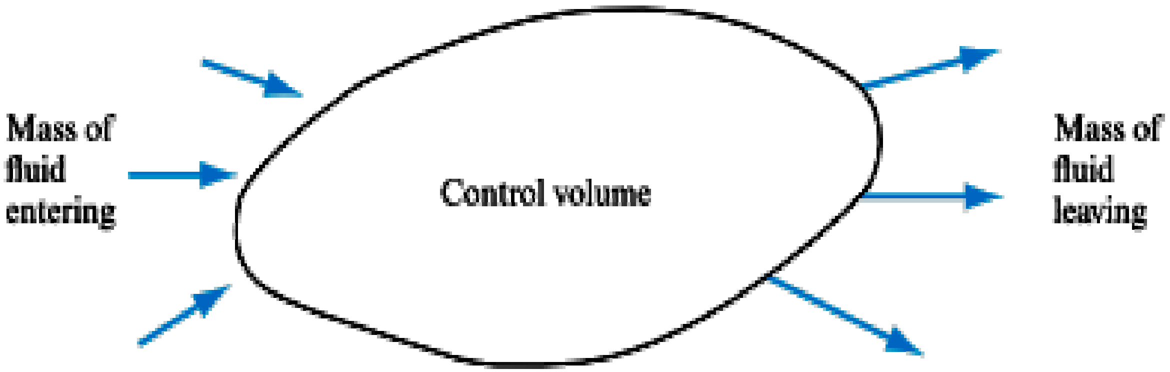
Continuity and Bernoulli's equation

Law of conservation of mass:

In a control volume **law of conservation of mass states** that ***the net mass flow through the volume will equal the mass stored or removed from the volume.*** Or, ***The net mass transfer to or from a control volume during a time interval Δt is equal to the net change (increase or decrease) in the total mass within the control volume during Δt.*** This principle of *conservation of mass* can be applied to a flowing fluid. Considering any fixed region in the flow constituting a control volume,

$$\boxed{\begin{pmatrix}\text{Total mass entering}\\ \text{the CV during } \Delta t\end{pmatrix} - \begin{pmatrix}\text{Total mass leaving}\\ \text{the CV during } \Delta t\end{pmatrix} = \begin{pmatrix}\text{Net change in mass}\\ \text{within the CV during } \Delta t\end{pmatrix}}$$

$$\boxed{\dot{m}_{in} - \dot{m}_{out} = \frac{\partial}{\partial t} m_{element}} \qquad \boxed{\int_{CV} \frac{\partial \rho}{\partial t}\, dV = \sum_{in} \dot{m} - \sum_{out} \dot{m}} \qquad \text{------- (1)}$$



$$\boxed{\frac{\partial}{\partial t} \int_{cv} \rho \, d\forall \approx \frac{\partial \rho}{\partial t} \delta x \, \delta y \, \delta z}$$

Consider a small control volume, stationary cubical element shown in Fig. 1

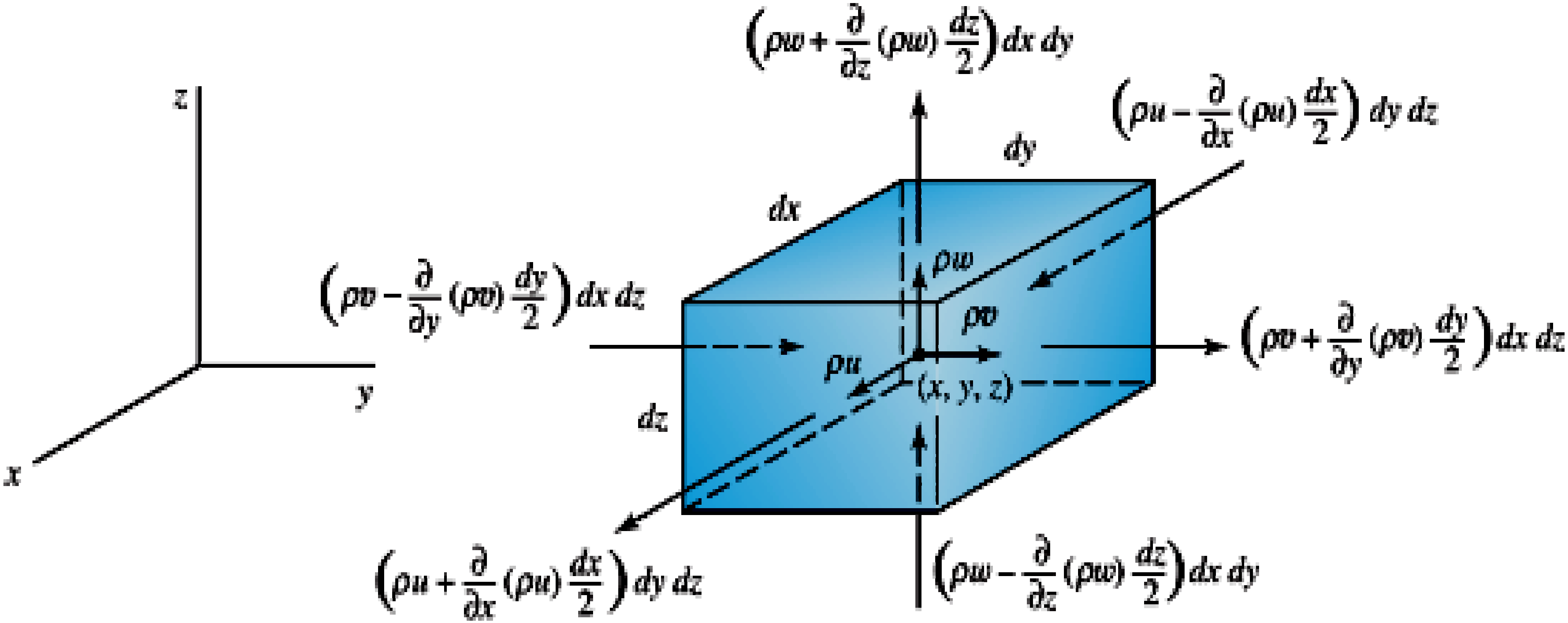


Fig.1: Infinitesimal control volume using Cartesian coordinates.

The rate of mass flow through the surfaces of the element can be obtained by ***considering the flow in each of the coordinate directions separately***. To perform this mass balance ***consider the mass rate of flow per unit area ρu, ρv, and ρw at the center of the element*** and then treat each of these quantities as a single variable. Note that we are really using a **Taylor series expansion of and neglecting higher order terms such as $(\delta x)^2$, $(\delta x)^3$ and so on**.

Then on the front face, $$\rho u|_{x+(\delta x/2)} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2} \qquad \text{------ (1a)}$$

and on the rear face, $$\rho u|_{x-(\delta x/2)} = \rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2} \qquad \text{------ (1b)}$$

When the right-hand sides of Eqs. (1a) and (1b) are ***multiplied by the area $\delta y \delta z$, the rate at which mass is crossing the front and rear sides of the element are obtained.***

Net mass flow rate into CV:

The left, bottom, and back faces contribute to mass ***inflow***.

$$\sum_{\text{in}} \dot{m} \cong \underbrace{\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\,dz}_{\text{rear face}} + \underbrace{\left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\,dz}_{\text{left face}} + \underbrace{\left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\,dy}_{\text{bottom face}}$$

Net mass flow rate out of CV:

Similarly, the right, top, and front faces contribute to mass ***outflow***.

$$\sum_{\text{out}} \dot{m} \cong \underbrace{\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\,dz}_{\text{front face}} + \underbrace{\left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\,dz}_{\text{right face}} + \underbrace{\left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\,dy}_{\text{top face}}$$

$$\begin{array}{c}\text{Net rate of mass} \\ \text{outflow in } x \text{ direction}\end{array} = \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z - \left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z = \frac{\partial(\rho u)}{\partial x}\delta x\,\delta y\,\delta z$$

------ (2a)

Similarly, $\begin{array}{c}\text{Net rate of mass} \\ \text{outflow in } y \text{ direction}\end{array} = \dfrac{\partial(\rho v)}{\partial y}\delta x\, \delta y\, \delta z$ ------ (2b)

and $\begin{array}{c}\text{Net rate of mass} \\ \text{outflow in } z \text{ direction}\end{array} = \dfrac{\partial(\rho w)}{\partial z}\delta x\, \delta y\, \delta z$ ------ (2c)

Thus, $\boxed{\begin{array}{c}\text{Net rate of} \\ \text{mass outflow}\end{array} = \left[\dfrac{\partial(\rho u)}{\partial x} + \dfrac{\partial(\rho v)}{\partial y} + \dfrac{\partial(\rho w)}{\partial z}\right]\delta x\, \delta y\, \delta z}$ ------ (2)

Since we considered in equ. (1) that
the net mass transfer = mass inflow – mass outflow, **so every terms of equ.(2) will have a negative sign.**

From Equ.(1), ***The differential equation that results from the conservation of mass.***

$$- \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\delta x\, \delta y\, \delta z = \frac{\partial}{\partial t}(\rho\, dx\, dy\, dz)$$

Subtracting the appropriate terms and dividing by *dx dy dz* yields

$$\boxed{\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0} \qquad \text{------ (3)}$$

This equation is also commonly referred to as **"the continuity equation"**.

Since density is considered a variable, we differentiate the products and put Eq. (3) in the form

$$\boxed{\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0} \qquad \text{------ (3)}$$

or, in terms of the substantial derivative

$$\boxed{\frac{D\rho}{Dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0}$$

Introducing the ***gradient operator***, called "**del,**" which, in rectangular coordinates, is

$$\nabla=\frac{\partial}{\partial x}\hat{\mathbf{i}}+\frac{\partial}{\partial y}\hat{\mathbf{j}}+\frac{\partial}{\partial z}\hat{\mathbf{k}}$$

The continuity equation can then be written in the form

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0}$$

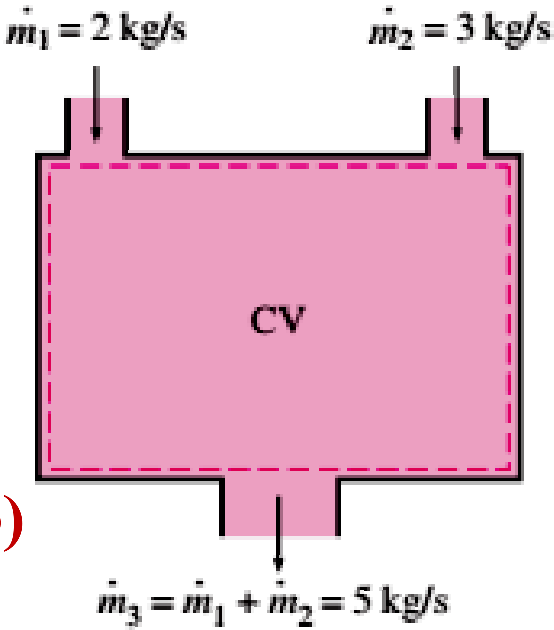
where $\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$; $\nabla \cdot \mathbf{V}$ is called the *divergence* of the velocity.

***The continuity equation is one of the fundamental equations of fluid mechanics** and **is valid for steady or unsteady flow, and compressible or incompressible fluids**.*

Special Cases:

(i) For steady flow of compressible fluids *$\delta/\delta t = 0$*

From equ. (1) $$\boxed{\sum_{\text{in}} \dot{m} = \sum_{\text{out}} \dot{m} \qquad (\text{kg/s})}$$ **------ (4a)**



It states that ***the total rate of mass entering a control volume is equal to the total rate of mass leaving it.***

From equ. (3) $$\boxed{\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$ **--- (4b)**

Many engineering devices such as ***nozzles, diffusers, turbines, compressors, and pumps*** **involve a single stream (only one inlet and one outlet)**. For these cases, we denote the inlet state by the subscript 1 and the outlet state by the subscript 2, and drop the summation signs. Then Eq. (4a) reduces, for ***single-stream steady-flow systems***, to

$$\boxed{\dot{m}_1 = \dot{m}_2 \quad \rightarrow \quad \rho_1 V_1 A_1 = \rho_2 V_2 A_2}$$

(ii) For both incompressible fluids (steady and unsteady)

The fluid density, ρ = constant

$\delta\rho/\delta t = 0, \quad \delta\rho/\delta x = 0, \quad \delta\rho/\delta y = 0, \delta\rho/\delta z = 0$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

Usually it is the case for liquids.

------- (5a)

$$\boxed{\nabla \cdot \mathbf{V} = 0} \qquad \boxed{\dot{V}_1 = \dot{V}_2 \rightarrow V_1 A_1 = V_2 A_2}$$

For two dimensional steady incompressible flow,

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

For one dimensional steady incompressible flow, $\delta u/\delta x = 0$

$$\frac{\partial(\rho u\, dy\, dz)}{\partial x} = 0$$

as $dy\ dz = dA$, Integrating **ρuA = constant.**

or, $\rho_1 u_1 A_1 = \rho_2 u_2 A_2$

Steady, incompressible flow (single stream $\rho_1 = \rho_2$):

$$\boxed{u_1 A_1 = u_2 A_2 = Q = \text{volumetric flow rate}}$$

Where A represents areas normal to the respective velocity vector.

Steady, incompressible flow:

$$\boxed{\sum_{\text{in}} \dot{V} = \sum_{\text{out}} \dot{V} \qquad (\text{m}^3/\text{s})}$$

Problem:

The velocity distribution for the flow of an incompressible fluid is given by $v_x = 3 - x$, $v_y = 4 + 2y$, $v_z = 2 - z$. Show that this satisfies the requirements of the continuity equation.

Solution:

For three-dimensional flow of an ***incompressible fluid (ρ = constant)***, the continuity equation simplifies to equation

$$\boxed{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0}$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

$$\frac{\partial v_x}{\partial x} = -1, \quad \frac{\partial v_y}{\partial y} = +2, \quad \frac{\partial v_z}{\partial z} = -1$$

and, hence, $$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = -1 + 2 - 1 = 0$$

which satisfies the requirement for continuity.

Problem:

Air flows in a pipe and the velocity at three neighboring points *A*, *B*, and *C*, 4 in. apart, is measured to be 274, 285, and 291 ft/sec, respectively, as shown in Fig. The temperature and pressure are 50°F and 50 psia, respectively, at point *B*. Approximate $d\rho/dx$ at that point, ***assuming steady, uniform flow***.

Solution:

$$\boxed{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0}$$

The continuity equation for this **steady ($\delta/\delta t = 0$)**, **uniform ($\delta/\delta y = \delta/\delta z = 0$)** flow reduces to

$$u\frac{d\rho}{dx} + \rho\frac{du}{dx} = 0$$

Use ordinary derivatives since u and ρ depend only on x. The velocity derivative is approximated by

$$\frac{du}{dx} \simeq \frac{\Delta u}{\Delta x} = \frac{291 - 274}{8/12} = 25.5\,\frac{\text{ft/sec}}{\text{ft}}$$

The density is

$$\rho = \frac{p}{RT} = \frac{50\ \text{lb/in}^2 \times 144\ \text{in}^2/\text{ft}^2}{1716\ \text{ft-lb/slug-°R} \times (50 + 460)\text{°R}} = 0.00823\ \text{slug/ft}^3$$

where absolute pressure and temperature are used. The density derivative is then approximated to be

$$\frac{d\rho}{dx} = -\frac{\rho}{u}\frac{du}{dx}$$

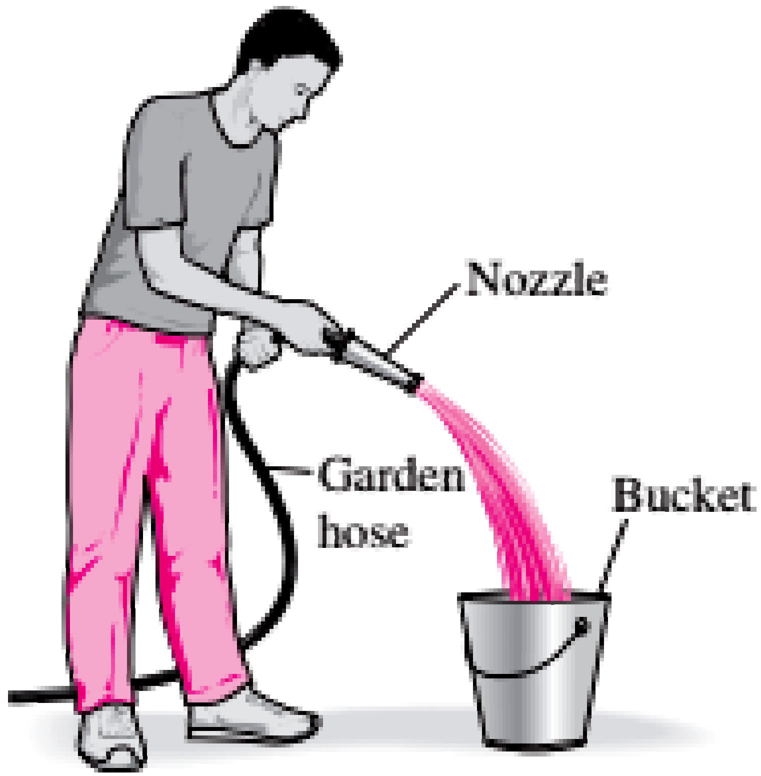
$$= -\frac{0.00823\ \text{slug/ft}^3}{285\ \text{ft/sec}} \times 25.5\ \text{sec}^{-1} = -0.000736\ \text{slug/ft}^4$$

Problem:

A garden hose attached with a nozzle is used to fill a **10-gal bucket**. The inner diameter of the hose is 2 cm, and it reduces to 0.8 cm at the nozzle exit as shown in fig. If it takes 50s to fill the bucket with water, determine

(*a*) the volume and mass flow rates of water through the hose, and

(*b*) The average velocity of water at the nozzle exit.



Solution:

Assumptions:

1. Water is an ***incompressible substance*** (i.e. **ρ = constant**).
2. Flow through the hose is ***steady*** (**$\delta/\delta t = 0$**).
3. There is ***no waste of water*** by splashing.

Properties:

Take the density of water, ρ_{water} = 1000 kg/m^3 = 1 kg/L.

Analysis:

(a) 10gal of water are discharged in 50 s, ***the volume and mass flow rates of water are***

$$\boxed{\begin{aligned}\dot{V} &= \frac{V}{\Delta t} = \frac{10\text{ gal}}{50\text{ s}}\left(\frac{3.7854\text{ L}}{1\text{ gal}}\right) = \mathbf{0.757\ L/s} \\ \dot{m} &= \rho\dot{V} = (1\text{ kg/L})(0.757\text{ L/s}) = \mathbf{0.757\ kg/s}\end{aligned}}$$

[*1 gallon (US) = 3.7854 liter*]

(b) The cross-sectional area of the nozzle exit is

$$A_e = \pi r_e^2 = \pi(0.4\text{ cm})^2 = 0.5027\text{ cm}^2 = 0.5027 \times 10^{-4}\text{ m}^2$$

The volume flow rate through the hose and the nozzle is constant.

Then the average velocity of water at the nozzle exit becomes

$$\boxed{V_e = \frac{\dot{V}}{A_e} = \frac{0.757\text{ L/s}}{0.5027 \times 10^{-4}\text{ m}^2}\left(\frac{1\text{ m}^3}{1000\text{ L}}\right) = \mathbf{15.1\ m/s}}$$

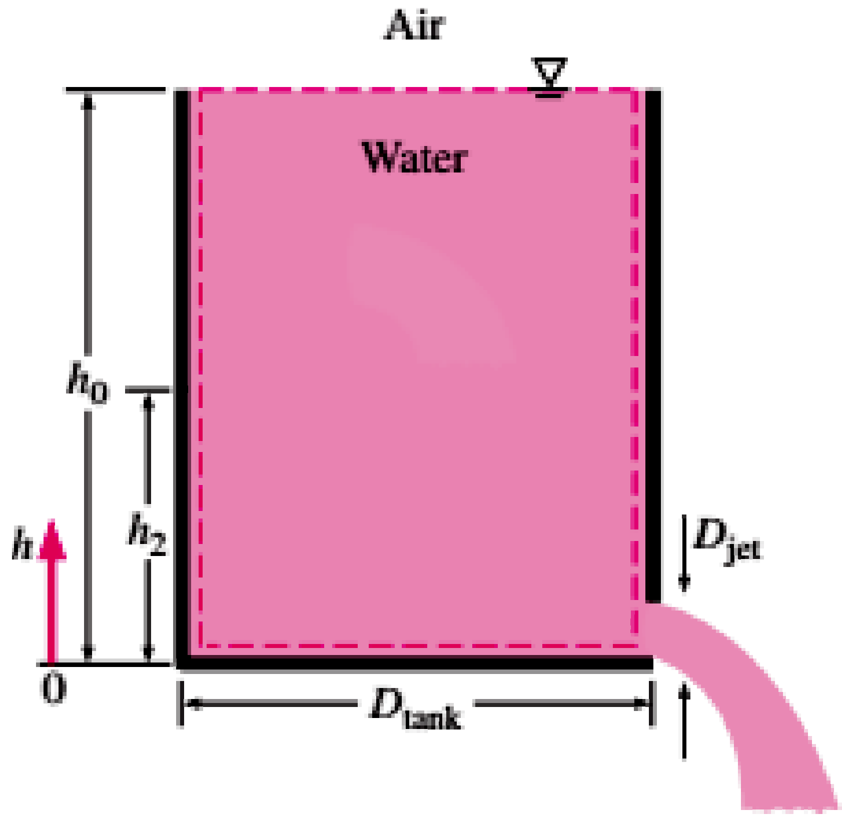
The cross-sectional area of the hose is, $A_h = \pi r^2_h = \pi\,(1\text{cm})^2 = 3.142 \times 10^{-4}\text{m}^2$

$$V_h = \frac{\dot{V}}{A_h} = \frac{0.757\ L/s}{3.142 \times 10^{-4}\ m^2}\left(\frac{1\ m^3}{1000\ L}\right) = 2.4\ m/s$$

The average velocity in the hose is 2.4 m/s. Therefore, ***the nozzle increases the water velocity by over six times.***

Problem:

A 4-ft-high, 3-ft-diameter cylindrical water tank whose top is open to the atmosphere is initially filled with water. Now the discharge plug near the bottom of the tank is pulled out, and a water jet whose diameter is 0.5 in streams out as shown in Fig. The average velocity of the jet is given by $V = \sqrt{2gh}$, where h is the height of water in the tank measured from the center of the hole (a variable) and g is the gravitational acceleration. Determine how long it will take for the water level in the tank to drop to 2 ft from the bottom.



Assumptions:

1. Water is an ***incompressible*** (**ρ = const**) substance.
2. The distance between the bottom of the tank and the center of the hole is negligible compared to the total water height.
3. The gravitational acceleration is 32.2 ft/s^2.

Analysis

Consider the volume occupied by water as the control volume. The size of the control volume decreases in this case as the water level drops, and thus this is a variable control volume. This is obviously an unsteady-flow problem since the properties (such as the amount of mass) within the control volume change with time.

The conservation of mass relation for a control volume undergoing any process is given in the rate form as

$$\boxed{\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}} \qquad \text{----- (1)}$$

During this process no mass enters the control volume ($\boxed{\dot{m}_{in} = 0}$), and the mass flow rate of discharged water can be expressed as

$$\dot{m}_{out} = (\rho VA)_{out} = \rho\sqrt{2gh}A_{jet} \qquad \text{----- (2)}$$

where $A_{jet} = \pi D^2_{jet}/4$ is the cross-sectional area of the jet, which is constant. Since the density of water is constant, the mass of water in the tank at any time is

$$m_{CV} = \rho V = \rho A_{tank}h \qquad \text{----- (3)}$$

where $A_{tank} = \pi D^2_{tank}/4$ is the base area of the cylindrical tank. Substituting Eqs. 2 and 3 into the mass balance relation (Eq. 1) gives

$$-\rho\sqrt{2gh}A_{\text{jet}} = \frac{d(\rho A_{\text{tank}}h)}{dt} \rightarrow -\rho\sqrt{2gh}(\pi D_{\text{jet}}^2/4) = \frac{\rho(\pi D_{\text{tank}}^2/4)\,dh}{dt}$$

Canceling the densities and other common terms and separating the variables gives

$$dt = -\frac{D_{\text{tank}}^2}{D_{\text{jet}}^2}\frac{dh}{\sqrt{2gh}}$$

Integrating from $t = 0$ at which $h = h_0$ to $t = t$ at which $h = h_2$ gives

$$\int_0^t dt = -\frac{D_{\text{tank}}^2}{D_{\text{jet}}^2\sqrt{2g}}\int_{h_0}^{h_2}\frac{dh}{\sqrt{h}} \rightarrow t = \frac{\sqrt{h_0}-\sqrt{h_2}}{\sqrt{g/2}}\left(\frac{D_{\text{tank}}}{D_{\text{jet}}}\right)^2$$

Substituting, the time of discharge is determined to be

$$t = \frac{\sqrt{4\text{ ft}}-\sqrt{2\text{ ft}}}{\sqrt{32.2/2\text{ ft/s}^2}}\left(\frac{3\times 12\text{ in}}{0.5\text{ in}}\right)^2 = 757\text{ s} = \mathbf{12.6\text{ min}}$$

Therefore, half of the tank will be emptied in 12.6 min after the discharge hole is unplugged.

Discussion

Using the same relation with ***$h_2 = 0$ gives $t = 43.1$min for the discharge of the entire amount of water in the tank***. Therefore, emptying the bottom half of the tank takes much longer than emptying the top half. This is due to the decrease in the ***average discharge velocity of water with decreasing h***.

Velocity and Acceleration:

By definition, **the velocity of a particle is *the time rate of change of the position vector for that particle.***

In Fig., the position of particle *A* relative to the coordinate system is given by its ***position vector***, r_A which (if the particle is moving) is a function of time. The time derivative of this position gives

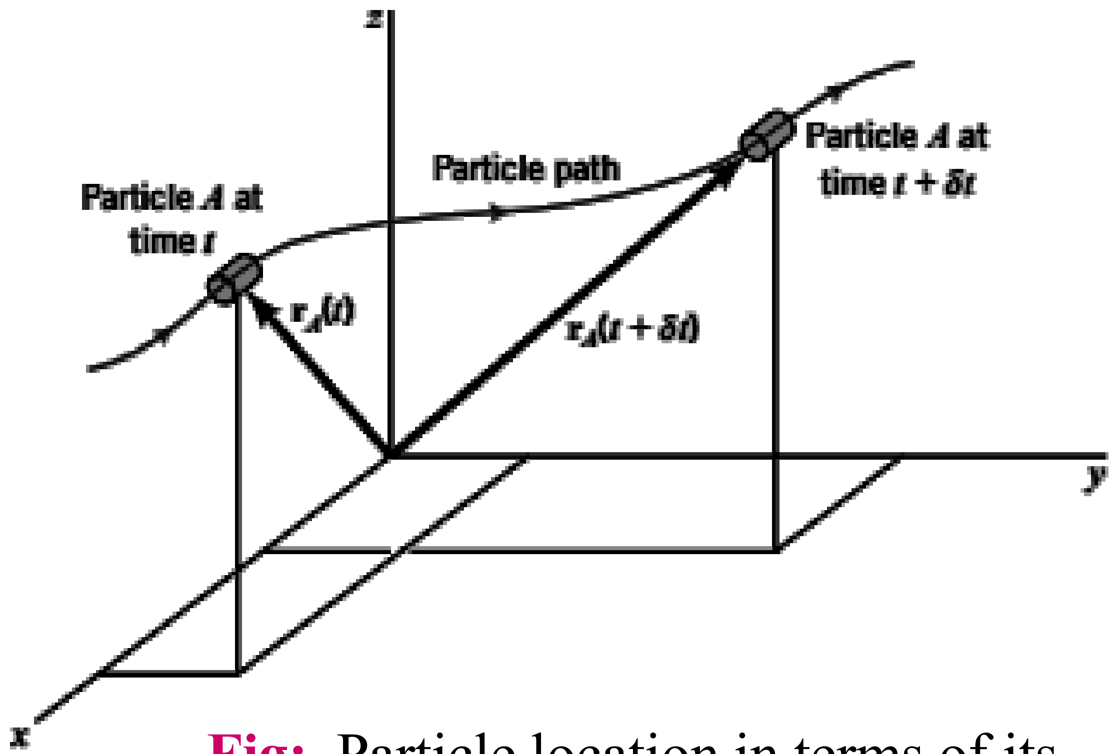


Fig: Particle location in terms of its position vector.

the velocity of the particle, $d\mathbf{r}_A/dt = \mathbf{V}_A$. The velocity for all of the particles that can be obtained in the field description of the velocity vector $\mathbf{V} = \mathbf{V}\ (x(t),\ y(t),\ z(t),\ t)$. The velocity vector $\mathbf{V}$ is given in component form as

$$\boxed{\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}}$$

where (u, v, w) are the velocity components in the x, y, and z directions, respectively, and $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors and

$$u = u(x,\ y,\ z\ ,\ t) \qquad v = v(x,\ y,\ z\ ,\ t) \qquad w = w(x,\ y,\ z\ ,\ t)$$

So,

$$\boxed{\mathbf{V} = u(x,\ y,\ z,\ t)\mathbf{i} + v(x,\ y,\ z,\ t)\mathbf{j} + w(x,\ y,\ z,\ t)\mathbf{k}}$$

Since the velocity is a vector, it has both a direction and a magnitude. ***The magnitude of*** $\mathbf{V}$, denoted

$$\boxed{V = |\mathbf{V}| = (u^2 + v^2 + w^2)^{1/2}}$$

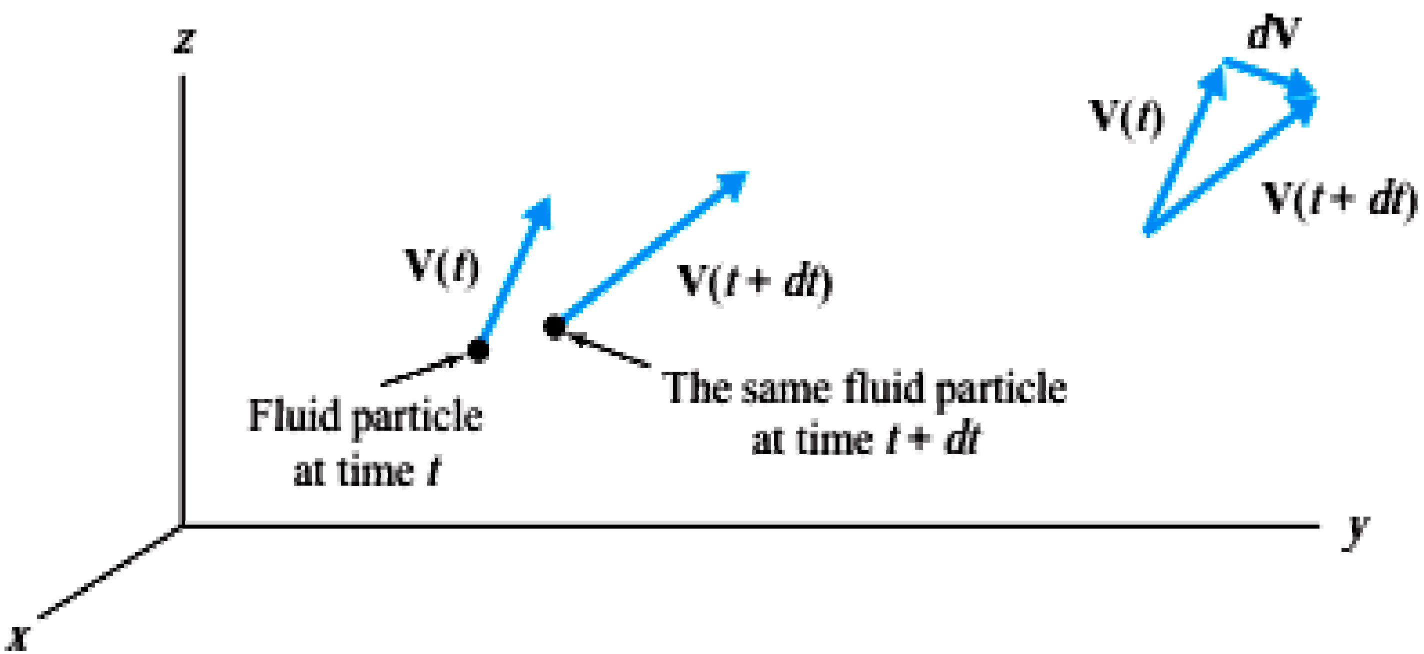
is ***the speed of the fluid***.

Two-dimensional flow: $\boxed{\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}}}$

One-dimensional flow: $\boxed{\mathbf{V} = u\hat{\mathbf{i}}}$

The acceleration of a particle is *the time rate of change of its velocity*. For unsteady flows the velocity at a given point in space (occupied by different particles) may vary with time, giving rise to a portion of the fluid acceleration. In addition, a fluid particle may experience an acceleration because its velocity changes as it flows from one point to another in space.

The acceleration of a fluid particle is found by considering a particular particle shown in Fig. Its velocity changes from $\mathbf{V}(t)$ at time t to $\mathbf{V}(t + dt)$ at time $t + dt$. The acceleration is, by definition,



$$\boxed{\mathbf{a} = \frac{d\mathbf{V}}{dt}}$$

Fig: Velocity of a fluid particle.

$$\boxed{\mathbf{V} = \mathbf{V}(x, y, z, t) = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}}$$

The quantity $d\mathbf{V}$ is, using the chain rule from calculus with $\mathbf{V} = \mathbf{V}(x, y, z, t)$

$$d\mathbf{V} = \frac{\partial \mathbf{V}}{\partial x}dx + \frac{\partial \mathbf{V}}{\partial y}dy + \frac{\partial \mathbf{V}}{\partial z}dz + \frac{\partial \mathbf{V}}{\partial t}dt$$

This gives the acceleration using as

$$\boxed{\mathbf{a} = \frac{d\mathbf{V}}{dt}} \qquad \mathbf{a} = \frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{V}}{\partial t}$$

Again $$\frac{dx}{dt} = u \qquad \frac{dy}{dt} = v \qquad \frac{dz}{dt} = w$$

The acceleration is then expressed as

$$\boxed{\mathbf{a} = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} + \frac{\partial \mathbf{V}}{\partial t}}$$

This is a vector result whose ***scalar components can be written as***

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

where a_x, a_y, and a_z are the x, y, and z components of the acceleration

The above result is often written in shorthand notation as $\boxed{\mathbf{a} = \frac{D\mathbf{V}}{Dt}}$
where, in Cartesian coordinates,

$$\frac{D(\ )}{Dt} \equiv \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z}$$

This derivative is called the **substantial derivative**, or **material derivative**. The ***time-derivative term*** on the right side of Eqs. ***is called the local derivative*** and the ***remaining terms on the right side*** in each equation form ***the convective / spatial derivative (V. ∇)***.

The gradient operator

$$\boxed{\nabla(\,) = {}^{\delta()}/_{\delta x}\,\hat{\imath} + {}^{\delta()}/_{\delta y}\,\hat{\jmath} + {}^{\delta()}/_{\delta z}\,\hat{k}}$$

Spatial derivative:

$$\boxed{\mathbf{V}\cdot\nabla(\ ) = u\partial(\ )/\partial x + v\partial(\ )/\partial y + w\partial(\ )/\partial z}$$

Consider a temperature $T = T(x, y, z, t)$ field associated with a given flow,

$$\boxed{\frac{dT_A}{dt} = \frac{\partial T_A}{\partial t} + \frac{\partial T_A}{\partial x}\frac{dx_A}{dt} + \frac{\partial T_A}{\partial y}\frac{dy_A}{dt} + \frac{\partial T_A}{\partial z}\frac{dz_A}{dt}}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + \mathbf{V}\cdot\nabla T$$

Flow Patterns:

Pathlines: A ***pathline*** is the actual path traversed by a given fluid particle or, **Path line is the trace of the path of a single particle over a period of time**. Path line ***shows the direction of the velocity of a particle at successive instants of time***. ***The pathline provides us with a "history" of the particle's locations***.

The pathline, or displacement of a particle, is defined by integration of the velocity components:

$$\boxed{x = \int u\,dt \qquad y = \int v\,dt \qquad z = \int w\,dt}$$

Given (u, v, w) as known functions of position and time, the integration is begun at a specified initial position (x_0, y_0, z_0, t_0). As shown in Fig.l(a), by recording on movie or video film a balloon released in the air, the path line can be observed. In general, **this is the curve in three-dimensional space.**

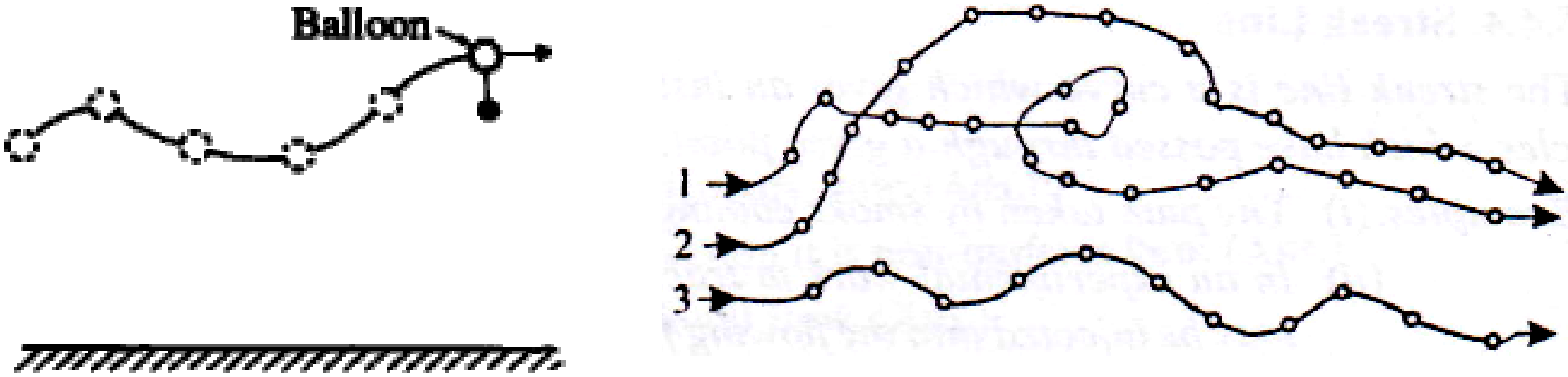


Fig.1(a) Pathline

Streak Line: The streak line is a curve which gives an instantaneous picture of the location of the fluid particles, which have passed through a given point.

Examples:

(i) The path taken by smoke coming out of chimney shown in Fig.1(b).

(ii) In an experimental work to trace the motion of fluid particles, a colored dye may be injected into the flowing fluid and the resulting colored filament lines at a given location give the steak lines shown in Fig.1(c).

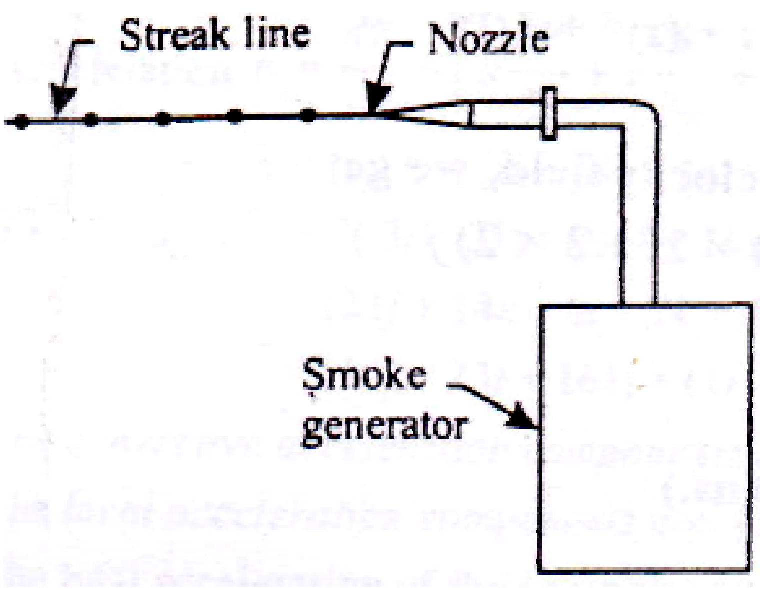


Fig.1(b): Streak line of smoke issuing from a nozzle.

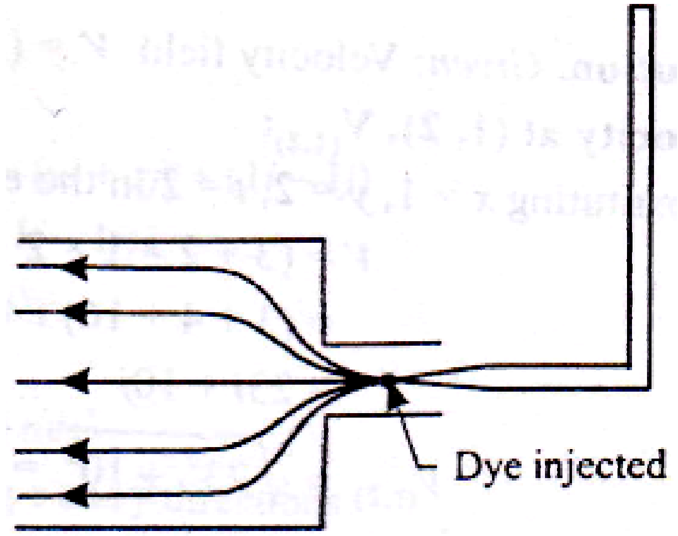
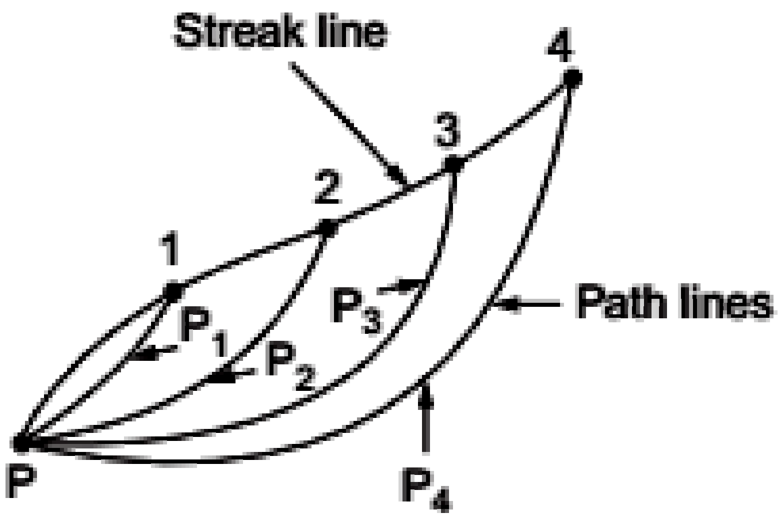


Fig.1(c): Streak lines at $t = t_1$



Particles P_1, P_2, P_3, P_4, starting from point P at successive times pass along path lines shown. At the instant of time considered the positions of the particles are at 1, 2, 3 and 4. A line joining these points is the streak line.

Streamlines: A streamline is a line everywhere tangent to the velocity vector at a given instant. Or, **A streamline may be defined on as an imaginary line within the flow so that the tangent at any point on it indicates the velocity at that point**. Since stream lines are tangent to the velocity vector at every point in the flow field, ***there can be no flow across a stream line.***

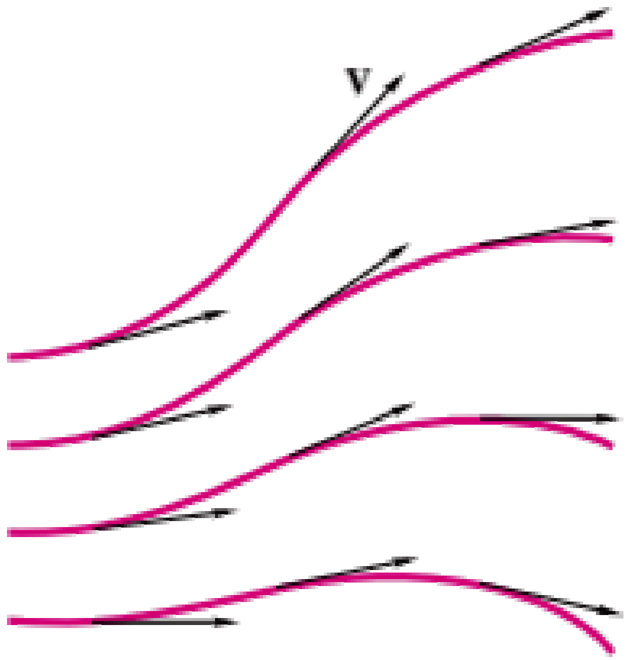


Fig.1(d): Streamlines

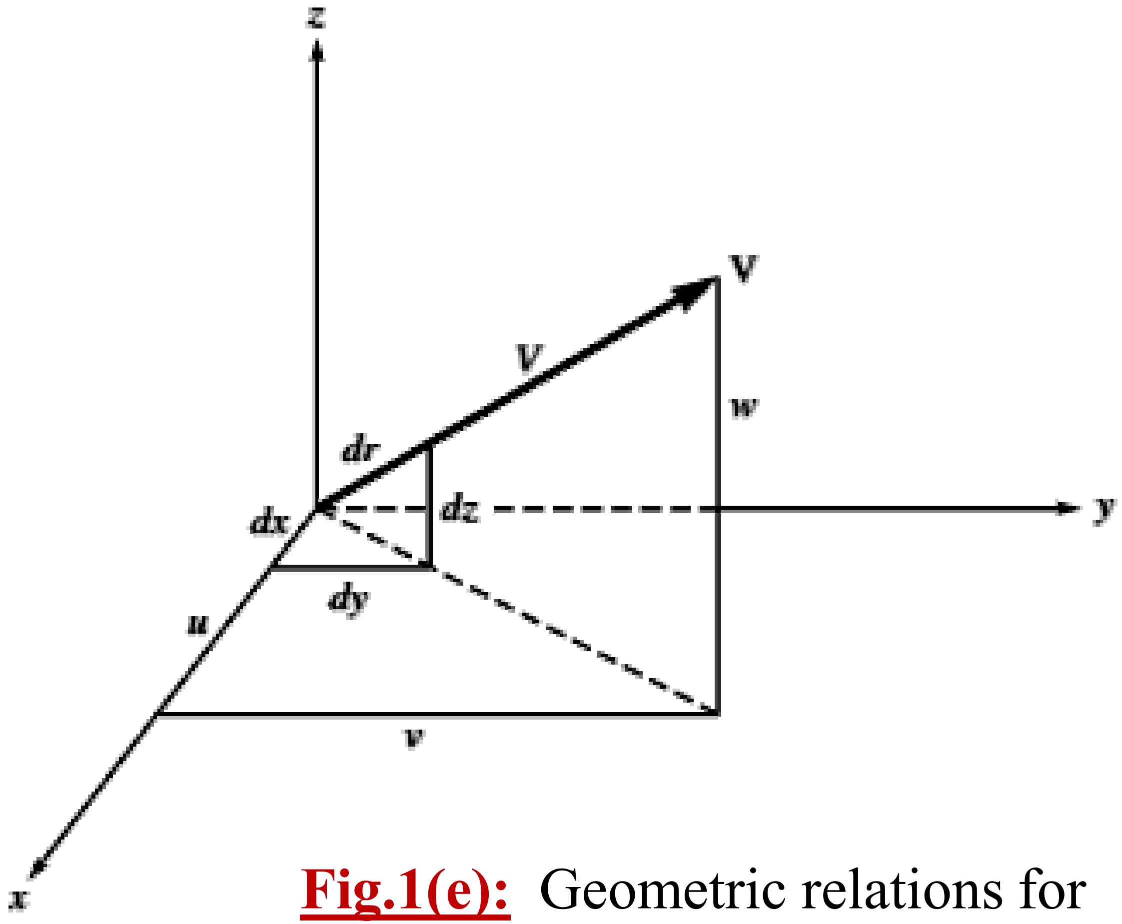


Fig.1(e): Geometric relations for defining a streamline.

Fig.1(e) shows an arbitrary velocity vector. If the elemental arc length *dr* of a streamline is to be parallel to **V**, their respective components must be in proportion:

$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dr}{V}}$$

The velocity vector of each particle occupying a point on the streamline is tangent to the streamline. An equation that expresses that the velocity vector is tangent to a streamline is

$$\boxed{\mathbf{V} \times d\mathbf{r} = 0}$$

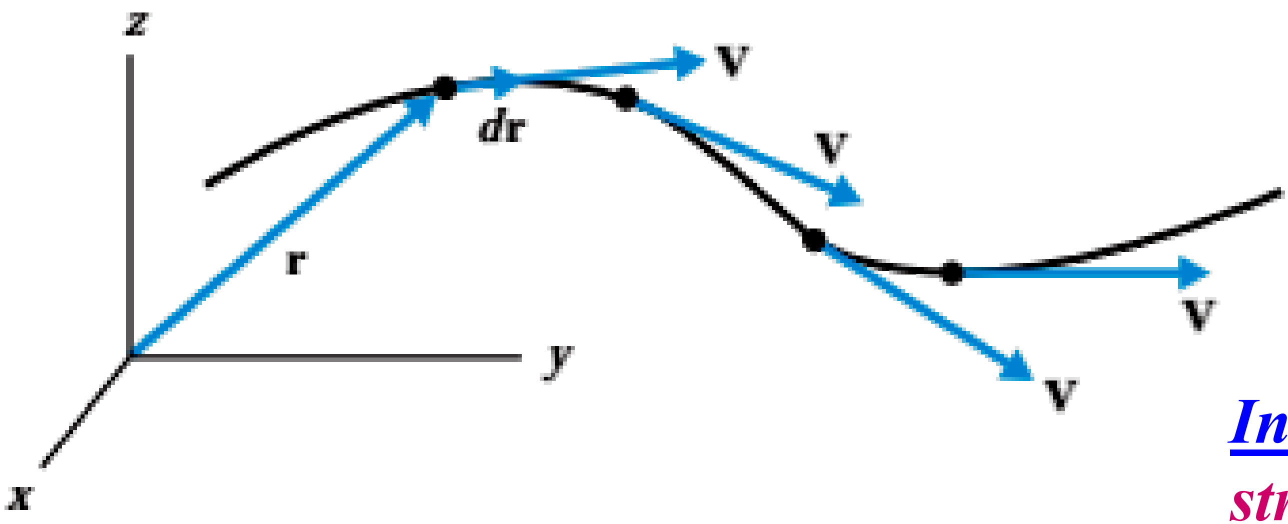
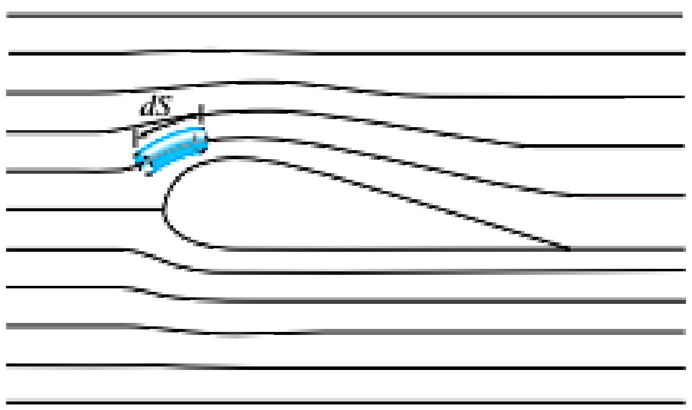


Fig: Streamline in a flow field.

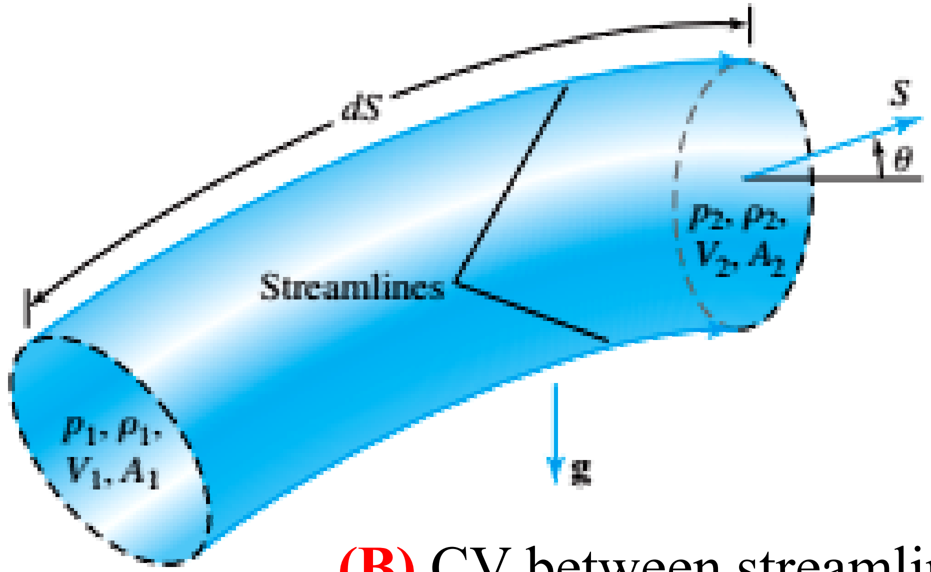
***In a steady flow*, pathlines, streaklines, and streamlines are all coincident.**

Euler's Equation (inviscid flow, $\mu = 0$) of Motion For Flow Along a Stream Line

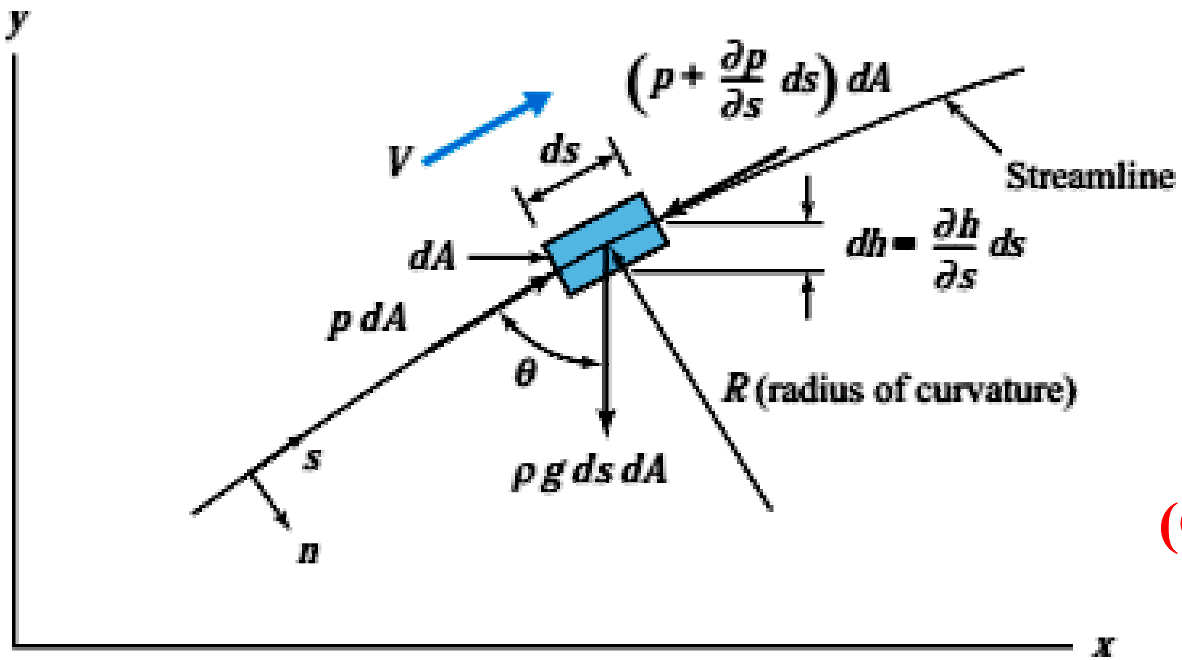
Consider a small element along the stream line, the direction being designated as *s*.



(A) Streamlines of flow over an airfoil.



(B) CV between streamlines.



(C) Particle moving along a streamline.

We wish to write the equations of motion in terms of ***the coordinate s***, ***distance along a streamline***, and ***the coordinate n***, ***distance normal to the streamline***. The pressure at the center of the fluid element is *p*. Newton's second law (which is referred to as the conservation of linear momentum relation in fluid mechanics).

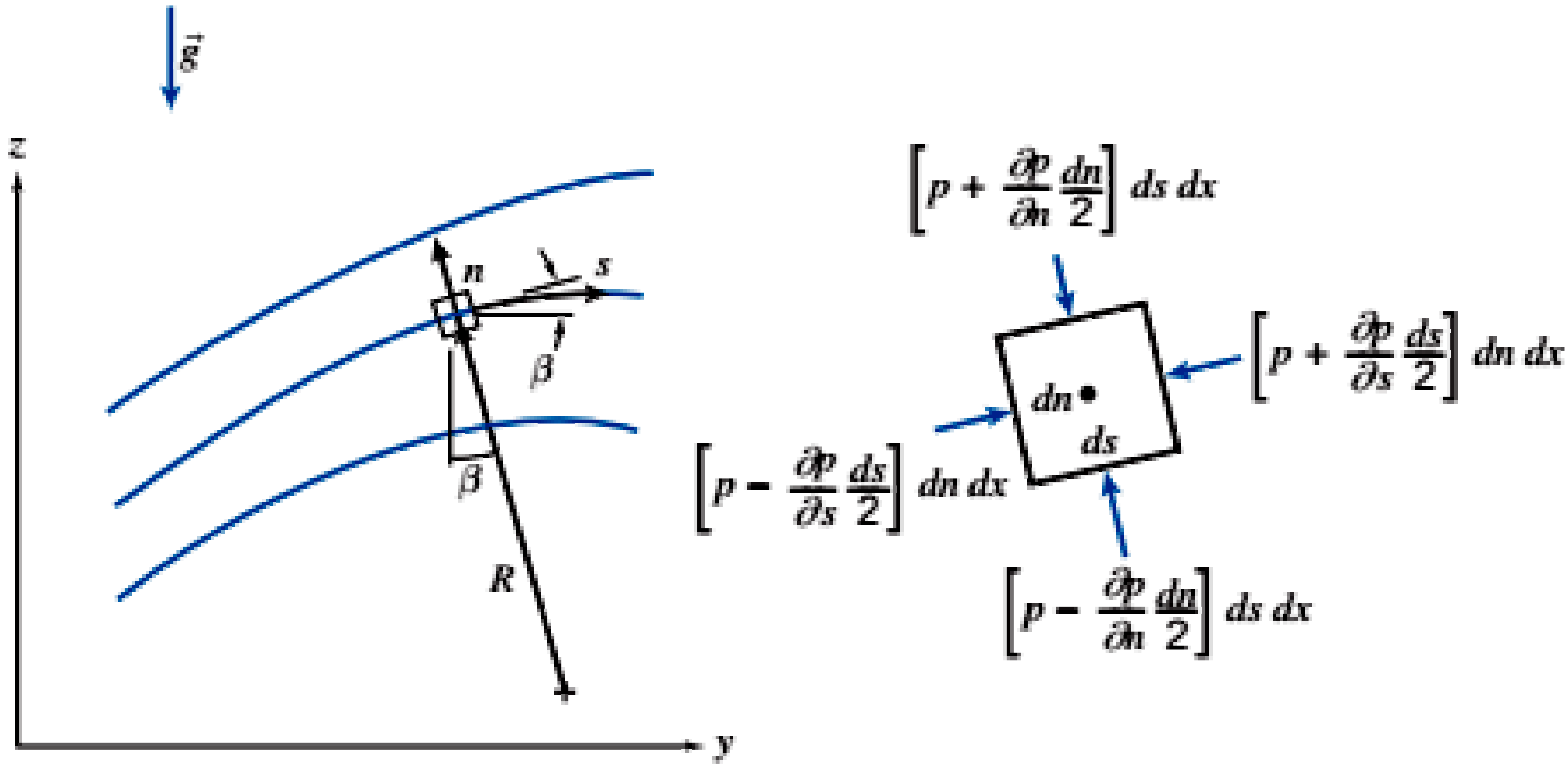


Fig 1: Fluid particle moving along a streamline.

■ ***Applying Newton's second law in the direction s of the streamline***, to the **fluid element of volume *ds dn dx***, then **neglecting viscous forces ($\mu = 0$)** we obtain

$$\left(p - \frac{\partial p}{\partial s}\frac{ds}{2}\right) dn\, dx - \left(p + \frac{\partial p}{\partial s}\frac{ds}{2}\right) dn\, dx - \rho g \sin\beta\, ds\, dn\, dx = \rho a_s\, ds\, dn\, dx$$

where ***β is the angle between the tangent to the streamline and the horizontal***, and ***a_s is the acceleration of the fluid particle along the streamline***. Simplifying the equation, we obtain

$$-\frac{\partial p}{\partial s} - \rho g \sin\beta = \rho a_s$$

Since $\sin\beta = \dfrac{\partial z}{\partial s}$ we can write $$-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} = a_s$$

Along any streamline $V = V(s, t)$, and the ***material or total acceleration of a fluid particle in the stream wise direction*** is given by

$$a_s = \frac{DV}{Dt} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}$$

Euler's equation in the streamwise direction with the *z* axis directed vertically upward is then

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial s}-g\frac{\partial z}{\partial s}=\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial s}} \qquad \text{----- (1)}$$

- For **steady flow**, and **neglecting body forces**, Euler's equation in the streamwise direction reduces to

$$\frac{1}{\rho}\frac{\partial p}{\partial s}=-V\frac{\partial V}{\partial s} \qquad \text{----- (1a)}$$

which indicates that (for **an incompressible, inviscid flow**) ***a decrease in velocity is accompanied by an increase in pressure and conversely***. This makes sense: The only force experienced by the particle is the net pressure force, so the ***particle accelerates toward low-pressure regions and decelerates when approaching high-pressure regions***.

- For **steady flow** ***$\partial V/\partial t = 0$***. Cancelling ∂s and using total derivatives in place of partials as these are independent quantities.

$$\frac{dp}{\rho} + gdz + VdV = 0 \qquad \text{----- (1b)}$$

This equation after dividing by *g*, is also written as,

$$\frac{dp}{\gamma} + d\left(\frac{V^2}{2g}\right) + dz = 0$$

Or,
$$\boxed{d\left[\frac{P}{\gamma} + \frac{V^2}{2g} + z\right] = 0} \qquad \text{----- (1c)}$$

which means that ***the quantity within the bracket remains constant along the flow***. **This equation is known as Euler's equation of motion**. The **assumptions** involved are:

1. Steady flow   2. Motion along a stream line and

3. Ideal fluid (frictionless, $\mu = 0$)

In the case on incompressible flow, ***this equation can be integrated to obtain Bernoulli's equation.***

■ To obtain **Euler's equation in a direction normal to the streamlines**, apply ***Newton's second law in the n direction to the fluid element***. Again, **neglecting viscous forces**, we obtain

$$\left(p - \frac{\partial p}{\partial n}\frac{dn}{2}\right) ds\, dx - \left(p + \frac{\partial p}{\partial n}\frac{dn}{2}\right) ds\, dx - \rho g \cos\beta\, dn\, dx\, ds = \rho a_n dn\, dx\, ds$$

where ***β is the angle between the n direction and the vertical***, and ***a_n is the acceleration of the fluid particle in the n direction***. Simplifying the equation, we obtain

$$-\frac{\partial p}{\partial n} - \rho g \cos\beta = \rho a_n$$

Since ***cosβ = δz/δn***, we write $$-\frac{1}{\rho}\frac{\partial p}{\partial n} - g\frac{\partial z}{\partial n} = a_n$$

The normal acceleration of the fluid element is toward the center of curvature of the streamline, in the minus *n* direction; thus in the coordinate system of Fig.1, the familiar centripetal acceleration is written

$$a_n = -\frac{V^2}{R}$$

For steady flow, where ***R is the radius of curvature of the streamline at the point***, Euler's equation normal to the streamline is written as

$$\frac{1}{\rho}\frac{\partial p}{\partial n} + g\frac{\partial z}{\partial n} = \frac{V^2}{R} \qquad \text{----- (2)}$$

For steady flow in a horizontal plane, Euler's equation normal to a streamline becomes

$$\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{V^2}{R} \qquad \text{----- (2a)}$$

Equation (2a) indicates that ***pressure increases in the direction outward from the center of curvature of the streamlines***. This also makes sense: Because the only force experienced by the particle is the net pressure force, ***the pressure field creates the centripetal acceleration***. In regions ***where the streamlines are straight, the radius of curvature, R, is infinite*** so ***there is no pressure variation normal to straight streamlines***.

Bernoulli Equation: *Integration of Euler's Equation Along a Streamline for Steady Flow*

Euler's equation for steady flow along a streamline (from Eq. 1b) is

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} = V\frac{\partial V}{\partial s} \qquad \text{----- (1b)}$$

If a fluid particle moves a distance, *ds*, along a streamline, then

$$\frac{\partial p}{\partial s}ds = dp \qquad \text{(the change in pressure along } s\text{)}$$

$$\frac{\partial z}{\partial s}ds = dz \qquad \text{(the change in elevation along } s\text{)}$$

$$\frac{\partial V}{\partial s}ds = dV \qquad \text{(the change in speed along } s\text{)}$$

Thus, after multiplying Eq. 1(b) by *ds*, we can write

$$-\frac{dp}{\rho} - g\,dz = V\,dV \qquad \text{Or,} \qquad \frac{dp}{\rho} + V\,dV + g\,dz = 0 \qquad \text{(along } s\text{)}$$

Integration of this equation gives (for compressible flow, ρ ≠ constant)

$$\boxed{\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant} \quad (\text{along } s)} \qquad \text{----- (3)}$$

The relationship between ρ and *p* must then be inserted for the given case. **For gases**, this will be of the form ***pρⁿ* = constant**, ***varying from adiabatic to isothermal conditions***, while, **for a liquid**, ***ρ(dp/dρ)* = *K***, the bulk modulus.

Before applying Eq. (3), specify the relation between pressure and density. **For the special case of incompressible flow**, ***ρ = constant***, and **Eq. (3) becomes the Bernoulli equation** after Daniel Bernoulli.

$$\boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)}} \qquad \text{----- (4)}$$

The terms represent energy per unit mass.

Dividing by ***g*,**

$$\boxed{p/\rho g + v^2/2g + z = \text{constant} = H,} \qquad \text{----- (4a)}$$

The terms represent the energy per unit weight.

$$\boxed{\frac{\rho v^2}{2} + p + \rho g z = \text{constant}} \qquad \text{----- (4b)}$$

The terms represent the energy per unit volume.

It is a simple algebraic equation for relating the pressure, velocity, and elevation in a fluid. These equations apply to a single streamline. ***The sum of the three terms is constant along any streamline, but the value of the constant may be different for different streamlines in a given stream***.

It is often convenient to write Eq. (4a) between two points (1) and (2) along a streamline and to express the equation in the "head" form by dividing each term by g so that

$$\boxed{\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2} \quad \boxed{\frac{p_2 - p_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) = 0} \qquad \text{----- (4c)}$$

$V^2/2$ as ***kinetic energy***, gz as ***potential energy***, and P/ρ as ***flow energy***, ***all per unit mass***. Therefore, the Bernoulli equation can be viewed as an expression of ***mechanical energy balance or, "conservation of mechanical energy principle."*** and can be stated as follows:

The sum of the kinetic, potential, and flow energies of a fluid particle is constant along a streamline during steady flow when the compressibility and frictional effects are negligible.

Recall that ***energy is transferred to a system as work when a force is applied to a system through a distance***. In the light of Newton's second law of motion, the Bernoulli equation can also be viewed as: ***The work done by the pressure and gravity forces on the fluid particle is equal to the increase in the kinetic energy of the particle.***

Note the **five assumptions**:

- –Inviscid flow (no shear stresses, $\mu = 0$)
- –Steady flow ($\partial V/\partial t = 0$)
- –Along a streamline ($a_s = V\, \partial V/ \partial s$)
- –Incompressible flow ($\rho = constant$, $\partial \rho/ \partial s = 0$, $\partial \rho/ \partial t = 0$)
- –Inertial reference frame ($\mathbf{A} = \mathbf{a}$)

Problem:

A liquid of specific gravity 1.3 flows in a pipe at a rate of 800 *l/s*, from point 1 to point 2 which is 1m above point 1. The diameters at section 1 and 2 are 0.6m and 0.3m respectively. If the pressure at section 1 is 10 bar, determine the pressure at section 2.

Solution:

Using Bernoulli equation: $\boxed{\dfrac{P}{\gamma} + z + \dfrac{V^2}{2g} = \text{constant}}$

***Steady, incompressible flow (single stream $\rho_1 = \rho_2$)*, the continuity equation is**

$$Q = AV = \textit{volumetric flow rate}$$

So, $V_1 = Q/A_1 = \{800\ (l/s) \times 1/1000\ (m^3/l)\} \div \{\pi \times 0.6^2/4\}$

$= 2.83$ m/s

Similarly, $V_2 = Q/A_2 = \{800\ (l/s) \times 1/1000\ (m^3/l)\} \div \{\pi \times 0.3^2/4\}$

$= 11.32$ m/s

$P_1 = 10$ bar $= 10 \times 10^5$ N/m^2 and

γ = sp. gravity x γ_{water} $= 1.3 \times 1000$ kg/m^3 x 9.81 m/s^2

$= 1.3 \times 9810$ N/m^3

Taking the datum as section 1, the pressure P_2 can be calculated.

$$\boxed{\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2}$$

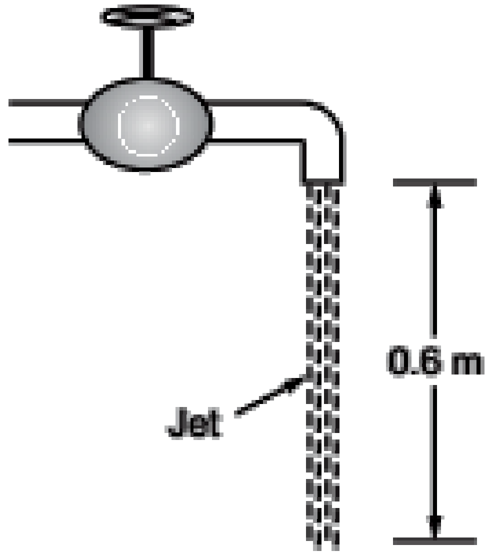
$$\frac{10 \times 10^5}{9810 \times 1.3} + 0 + \frac{2.83^2}{2 \times 9.81} = \frac{P_2}{9810 \times 1.3} + 1 + \frac{11.32^2}{2 \times 9.81}$$

Solving, **$P_2 = 9.092$ bar (9.092×10^5 N/m^2) Ans.**

Problem:

A tap discharges water evenly in a jet at a velocity of 2.6 m/s at the tap outlet, the diameter of the jet at this point being 15 mm. The jet flows down vertically in a smooth stream. Determine the velocity and the diameter of the jet at 0.6 m below the tap outlet.

Solution:



The pressure around the jet is atmospheric throughout. Taking ***the tap outlet as the datum (point 1)*** and ***the position 0.6 m below the tap outlet as point 2***.

So, $P_1 = 1 \text{ atm} = P_2$ and $z_1 = 0$

$z_2 = -0.6\text{m}$ and $V_1 = 2.6 \text{ m/s}$

Using Bernoulli equation.

$$\boxed{\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2}$$

$$\frac{2.6^2}{2 \times 9.81} = -0.6 + \frac{V_2^2}{2 \times 9.81}$$

$V_2 = 4.3$ m/s Ans.

***Steady, incompressible flow (single stream $\rho_1 = \rho_2$)*, the continuity equation is**

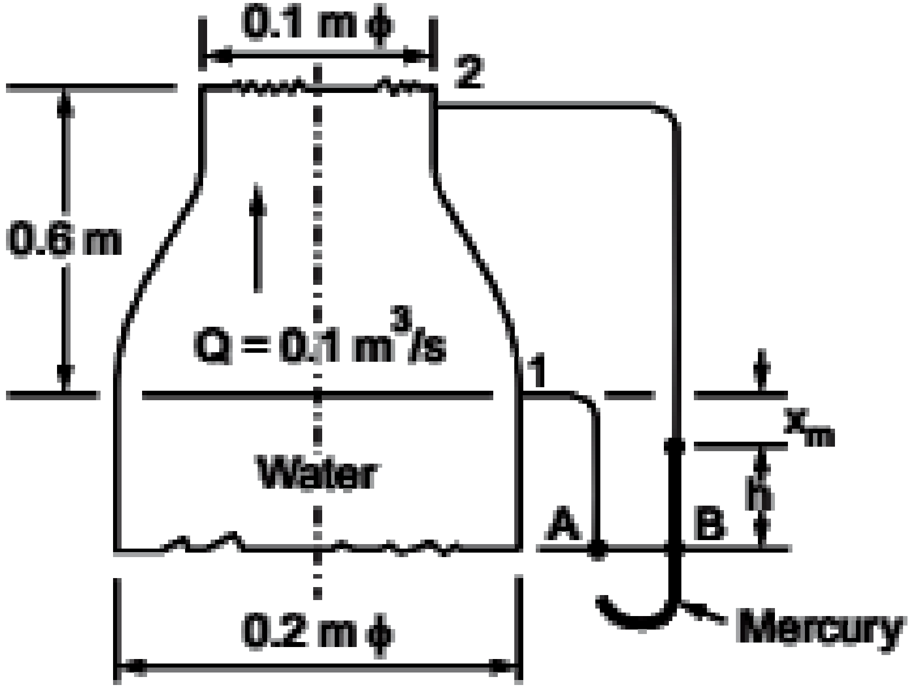
$$Q = AV = A_1V_1 = A_2V_2$$

$$\frac{\pi \times 015^2}{4} \times 2.6 = \frac{\pi \times D^2}{4} \times 4.3$$

$\therefore D_2 = 0.01166$ m or 11.66 mm Ans.

Problem:

Water flows in a tapering pipe vertically as shown in Fig. *Determine the manometer reading "h"*. The manometer fluid has a specific gravity of 13.6. The flow rate is 100 l/s.



Solution:

For ***Steady, incompressible flow (single stream $\rho_1 = \rho_2$)*, the continuity equation is**

$$Q = AV = \text{volumetric flow rate}$$

So, $V_1 = Q/A_1 = 0.1\ (m^3/s) \div \{\pi \times 0.2^2/4\} = 3.183\ m/s$

Similarly, $V_2 = Q/A_2 = 0.1\ (m^3/s) \div \{\pi \times 0.1^2/4\} = 12.732\ m/s$

It is desired to determine $P_1 - P_2$. Rearranging Bernoulli equation for this flow

$$\boxed{\frac{P_1 - P_2}{\gamma} = (z_2 - z_1) + \frac{V_2^2 - V_1^2}{2g}}$$

$$\frac{P_1 - P_2}{\gamma} = 0.6 + (12.732^2 - 3.183^2)/(2 \times 9.81) = 8.346 \text{ m of water}$$

For water $\gamma = 9810$ N/m^3. For the manometer configuration, considering the level AB and equating the pressures at A and B

$$\frac{P_1}{\gamma} + x + h = \frac{P_2}{\gamma} + 0.6 + x + sh$$

(where x, h are shown on the diagram and s is specific gravity).

$$\frac{P_1 - P_2}{\gamma} = 0.6 + h(s - 1), \text{ substituting the values,}$$

$$8.346 = 0.6 + h(13.6 - 1)$$

***h* = 0.6148 m or 61.48 cm   Ans.**

Static, Stagnation, and Dynamic Pressures

$$\boxed{\frac{v^2}{2g} + \frac{p}{\rho g} + z = H = \text{constant}} \qquad \boxed{\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2}$$

----- (4a) ----- (4c)

The terms of above eqn. represent energy per unit weight, and ***they have the units of length (m)*** so ***they are commonly termed heads***.

$\frac{v^2}{2g}$: velocity head

$\frac{p}{\rho g}$: pressure head

z : potential head

H : total head

The sum of the two terms ***(p/g + z)*** is called the ***piezometric head*** and ***the sum of the three terms*** the ***total head***.

$$\boxed{\frac{\rho v^2}{2} + p + \rho g z = \text{constant}}$$

----- (4b)

The terms represent the energy per unit volume.

Each term in this equation has pressure units, and thus ***each term represents some kind of pressure:***

- ***P*** is the **static pressure** (it does not incorporate any dynamic effects); it represents the actual thermodynamic pressure of the fluid. This is the same as the pressure used in thermodynamics and property tables.

- $\rho V^2/2$ is the **dynamic pressure**; it represents the pressure rise when the fluid in motion is brought to a stop isentropically.

- $\rho g z$ is the **hydrostatic pressure**, which is not pressure in a real sense since its value depends on the reference level selected; it accounts for the elevation effects, i.e., of fluid weight on pressure.

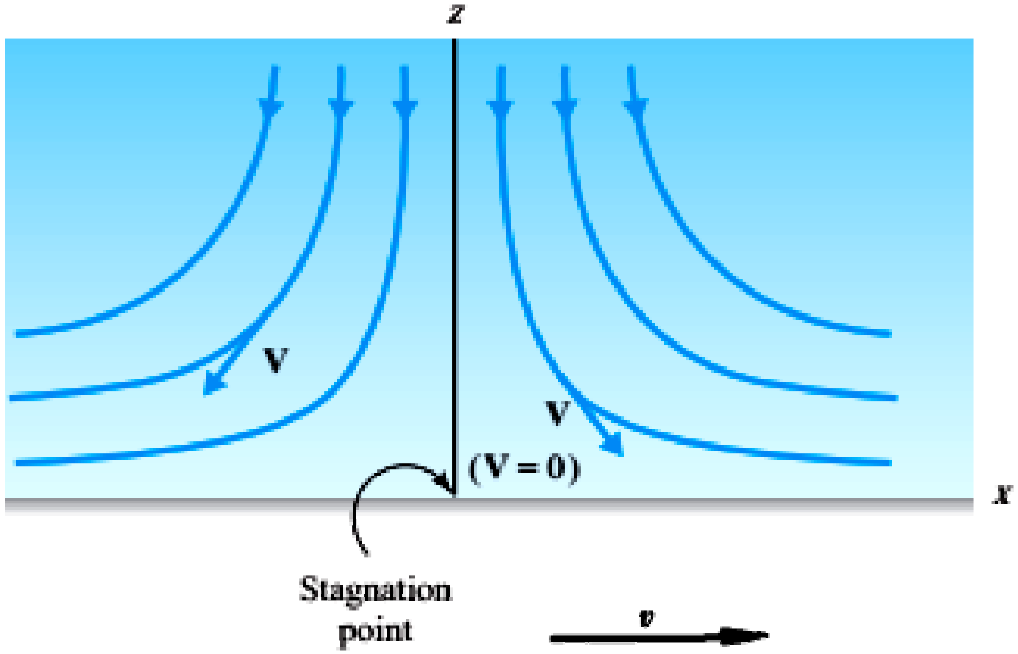
The sum of the static, dynamic, and hydrostatic pressures is called the total pressure. Therefore, **the Bernoulli equation states** that **the total pressure along a streamline is constant.**

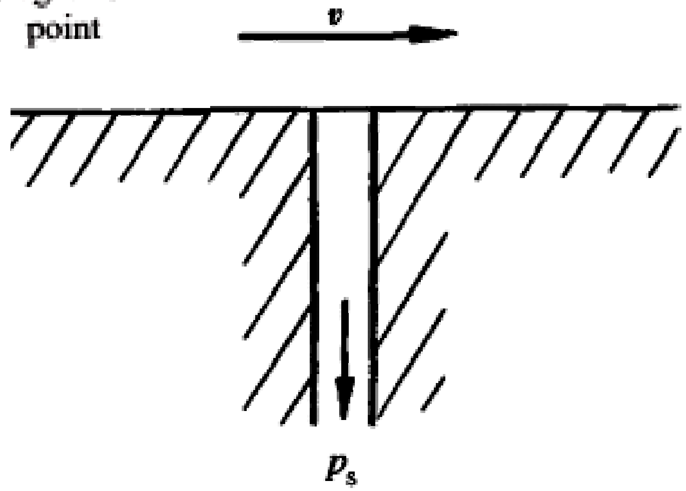
- The ***sum of the static and dynamic pressures*** is called the **stagnation pressure (total pressure)**, the pressure at a stagnation point in the flow. and it is expressed as

$$\boxed{P_{stag} = P + \rho \frac{V^2}{2}}$$

The stagnation pressure represents the pressure at a point ***where the fluid is brought to a complete stop isentropically (frictionless process).***

Static pressure p_s can be detected, as shown in Fig., ***by punching a small hole (pressure tap) vertically in the solid wall face parallel to the flow.*** Static pressure probes, are available commercially in sizes as small as 1.5 mm (1/16 in.) in diameter.





For incompressible flow, the Bernoulli equation can be used to relate changes in speed and pressure along a streamline for such a process.

$$\boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)}} \qquad \text{----- (4)}$$

Neglecting elevation differences, Eq. (4) becomes $\frac{p}{\rho} + \frac{V^2}{2} = \text{constant}$

If the static pressure is p at a point in the flow where the speed is V, then the stagnation pressure, $p_0 = p_{stag}$, where the stagnation speed, V_0, is zero, may be computed from

$$\frac{p_0}{\rho} + \underbrace{\cancel{\frac{V_0^2}{2}}}_{=0} = \frac{p}{\rho} + \frac{V^2}{2}$$

Or,

$$\boxed{p_0 = p + \frac{1}{2}\rho V^2}$$

The static, dynamic, and stagnation pressures are shown in Fig. ***When static and stagnation pressures are measured at a specified location***, the **fluid velocity at that location** can be calculated from

$$\boxed{V = \sqrt{\frac{2(P_{stag} - P)}{\rho}}}$$

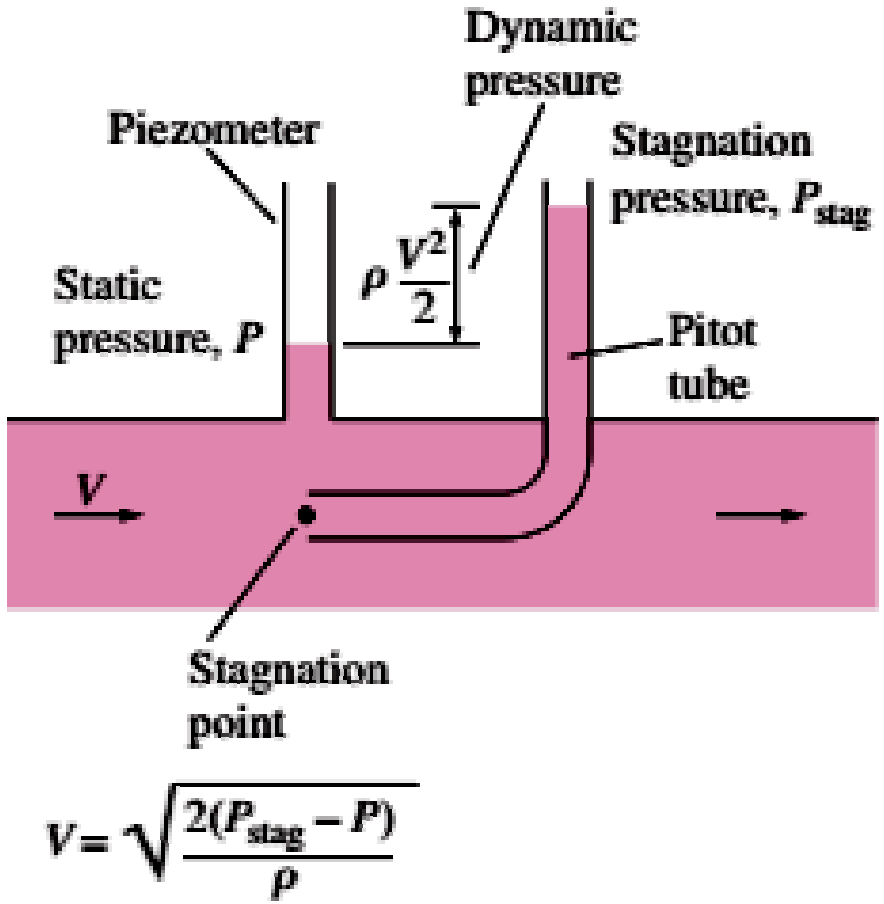


Fig: The static, dynamic, and stagnation pressures.

A **static pressure tap** is simply a ***small hole drilled into a wall*** such that ***the plane of the hole is parallel to the flow direction***. It measures the static pressure.

A **Pitot tube** is a ***small tube with its open end aligned into the flow*** so as ***to sense the full impact pressure of the flowing fluid***. It **measures** the **stagnation pressure**.

In situations ***in which the static and stagnation pressure of a flowing liquid are greater than atmospheric pressure***, a vertical transparent tube called a **piezometer tube** (or simply **a piezometer**) can be attached to the pressure tap and to the Pitot tube, as sketched in previous Fig. ***The liquid rises in the piezometer tube to a column height (head) that is proportional to the pressure being measured***. **If the pressures to be measured are below atmospheric, or** if **measuring pressures in gases**, **piezometer tubes do not work**.

However, the ***static pressure tap and Pitot tube*** can still be used, but they ***must be connected to some other kind of pressure measurement device*** such as a **U-tube manometer** or a **pressure transducer**.

A **pitotstatic probe** is also used to measure the difference between total and static pressure with one probe as shown in following fig.

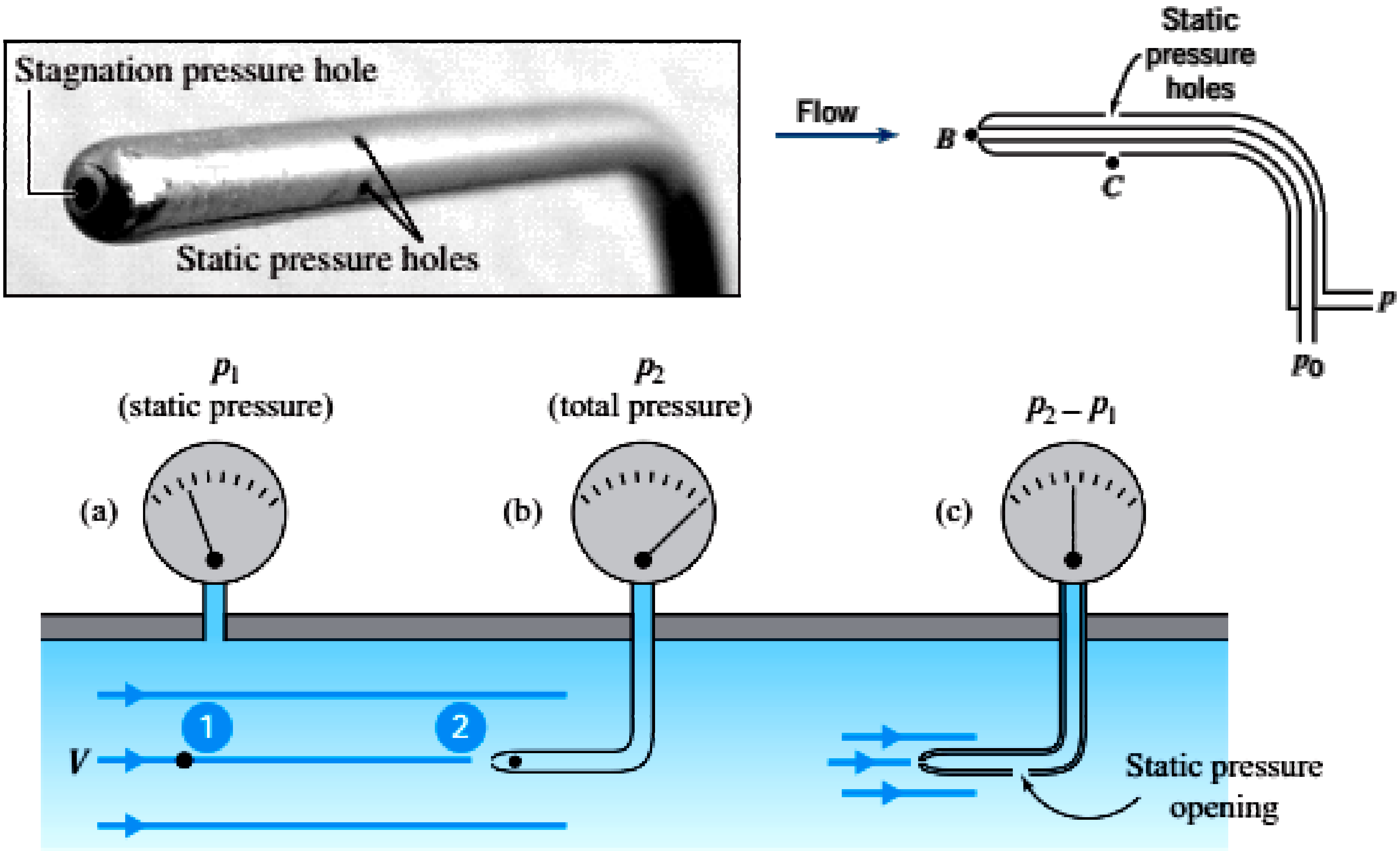
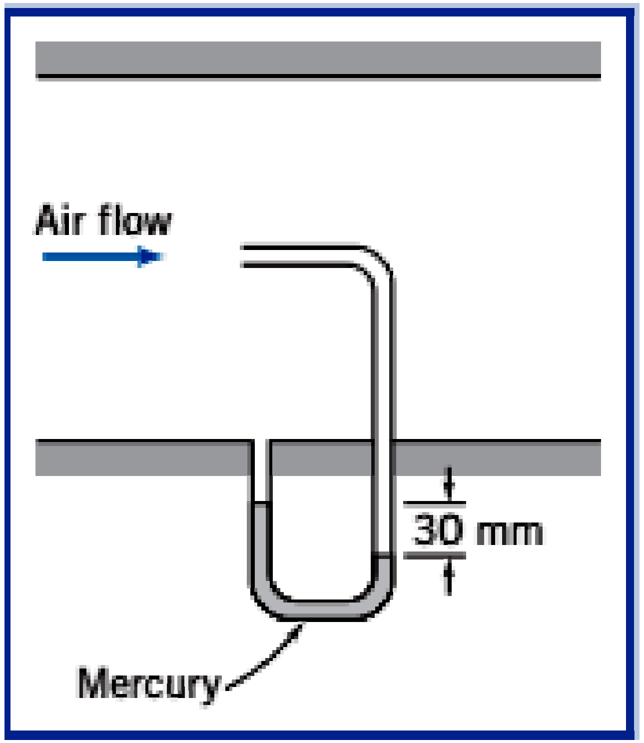


Fig. Pressure probes: (a) piezometer; (b) pitot probe; (c) pitot-static probe.

Problem:

A pitot tube is inserted in an air flow (at STP) to measure the flow speed. The tube is inserted so that it points upstream into the flow and the pressure sensed by the tube is the stagnation pressure. The static pressure is measured at the same location in the flow, using a wall pressure tap. If the pressure difference is 30 mm of mercury, determine the flow speed.



Solution:

Governing equation:

$$\boxed{\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)}}$$

Assumptions: (1) Steady flow.

(2) Incompressible flow.

(3) Flow along a streamline.

(4) Frictionless deceleration along stagnation streamline.

Bernoulli's equation along the stagnation streamline (with $\Delta z = 0$) yields

$$\frac{p_0}{\rho} = \frac{p}{\rho} + \frac{V^2}{2}$$

p_0 is the stagnation pressure at the tube opening where the speed has been reduced, without friction, to zero. Solving for V gives

$$V = \sqrt{\frac{2(p_0 - p)}{\rho_{\text{air}}}}$$

From the diagram, $p_0 - p = \rho_{\text{Hg}} g h = \rho_{\text{H}_2\text{O}} g h SG_{\text{Hg}}$

$$V = \sqrt{\frac{2\rho_{\text{H}_2\text{O}} g h SG_{\text{Hg}}}{\rho_{\text{air}}}}$$

$$= \sqrt{2 \times 1000 \frac{\text{kg}}{\text{m}^3} \times 9.81 \frac{\text{m}}{\text{s}^2} \times 30\text{ mm} \times 13.6 \times \frac{\text{m}^3}{1.23\text{ kg}} \times \frac{1\text{ m}}{1000\text{ mm}}}$$

V = 80.8 m/s **Ans.**

At $T = 20^0$C, the speed of sound in air is 343 m/s. Hence, Mach number, M = 0.236 and the assumption of incompressible flow is valid.

Mach Number, M:

In fluid dynamics, the Mach number (M or Ma) is a dimensionless quantity representing the ratio of flow velocity past a boundary to the local speed of sound.

$$M = V/c$$

= local flow velocity w.r.t. boundary / speed of sound in the medium

Type of Flow	Mach Number, M
Incompressible Flow	$M < 0.3$
Compressible Subsonic flow	$0.3 < M < 1$
Transonic flow	M ranging between values less than and more than 1(say 0.8-1.2)
Supersonic flow	$1 < M < 7$
Hypersonic flow	$M > 7$

STP (at sea level): $T = 15^0C$ (288.15K), $P = 101.325$ kPa,
$\rho = 1.225$ kg/m,^{3} $\mu = 1.79 \times 10^{-3}$ Ns/m,^{2} sonic velocity of air = 340.29 m/s

Bernoulli equation:

$$\boxed{\frac{P}{\gamma} + z + \frac{V^2}{2g} = \text{constant}}$$

$$\boxed{\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2} \quad \text{--- (4c)}$$

Assume ***the pipe line is horizontal***, and $z_1 = z_2$ in eqn (4c)

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho g} = \frac{v_2}{2g} + \frac{p_2}{\rho g}$$

Also, from the ***continuity equation***,

$$v_1 A_1 = v_2 A_2$$

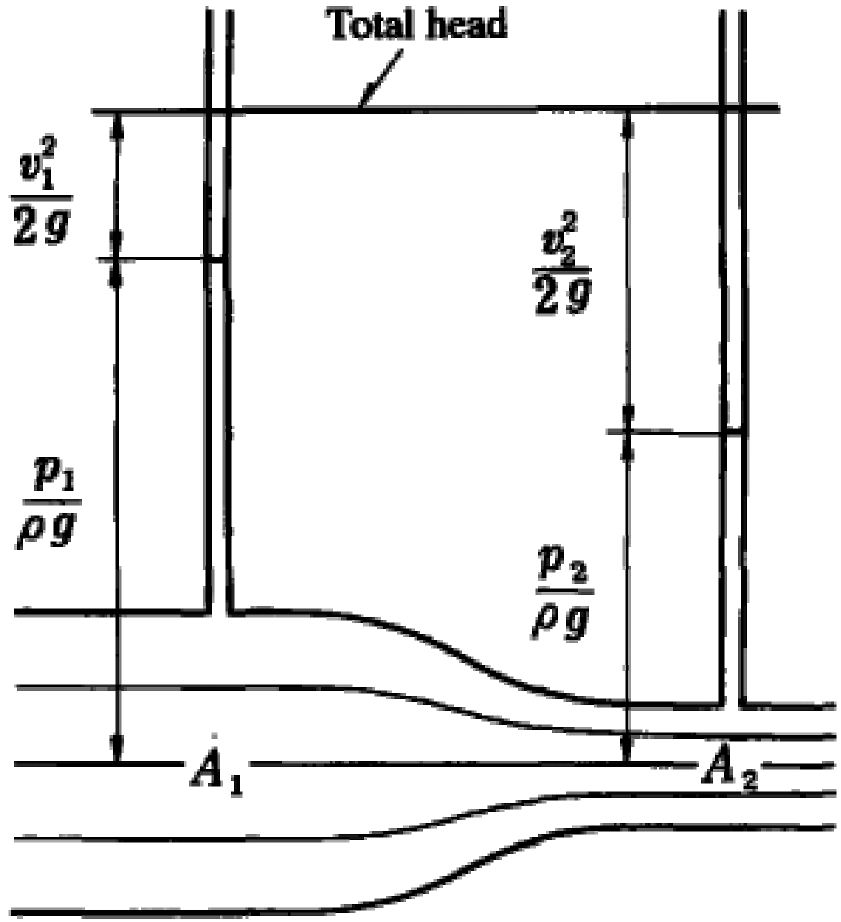


Fig: Exchange between pressure head and velocity head

Consequently, whenever **$A_1 > A_2$, then $v_1 < v_2$ and $p_1 > p_2$** . In other words, where the flow channel is narrow (where the streamlines are dense), the flow velocity is large and the pressure head is low.

Hydraulic Grade Line (HGL) and Energy Grade Line (EGL)

For a steady, incompressible, frictionless flow, the Bernoulli equation is:

$$\boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)}}$$

The three terms comprised of "***pressure," kinetic, and potential energies to make up the total mechanical energy per unit mass of the fluid***.

Again,

$$\boxed{\frac{v^2}{2g} + \frac{p}{\rho g} + z = H = \text{constant}}$$

- Here H is the ***total head of the flow***; it ***measures the total mechanical energy in units of meters or feet***. **In a real fluid** (***one with friction***) ***this head*** will not be constant but ***will continuously decrease in value as mechanical energy is converted to thermal***; in this Lecture H is constant.

- We can define this H to be the **energy grade line (EGL)**. Energy line is the plot of $P/\gamma + z + V^2/2g$ along the flow. **It is constant along the flow when losses are negligible.**

$$\boxed{EGL = \frac{P}{\rho g} + \frac{V^2}{2g} + z} \quad \text{--- (5a)}$$

- The plot of $P/\gamma + z$ along the flow is called the **hydraulic grade line (HGL).** ***When velocity increases this will dip and when velocity decreases this will rise***.

$$\boxed{HGL = \frac{p}{\rho g} + z} \quad \text{--- (5b)}$$

- The hydraulic gradient line provides useful information about pressure variations (static head) in a flow. The ***difference between the energy line and hydraulic gradient line gives the value of*** **dynamic head (velocity head).**

$$\boxed{EGL - HGL = \frac{V^2}{2g}} \quad \text{--- (5c)}$$

Example-1: The graphical interpretation of the EGL and HGL, refer to the example shown in Fig. which shows frictionless flow from a reservoir, through a pipe reducer.

- At all locations the EGL is the same because there is no loss of mechanical energy.

- Station 1 is at the reservoir, and here the EGL and HGL coincide with the **free surface**: in Eqs. (5a) and (5b) ***P* = 0 (gage), *V* = 0, and *z* = *z*₁**, so ***EGL*₁ = *HGL*₁ = *H* = *z*₁**; ***all of the mechanical energy is potential***. (If we were to place a pitot tube in the fluid at station 1, the fluid would of course just rise to the free surface level.)

- At station 2 we have a pitot (total head) tube and a static head tap. The pitot tube's column indicates the correct value of the EGL ($EGL_1 = EGL_2 = H$), but something changed between the two stations:

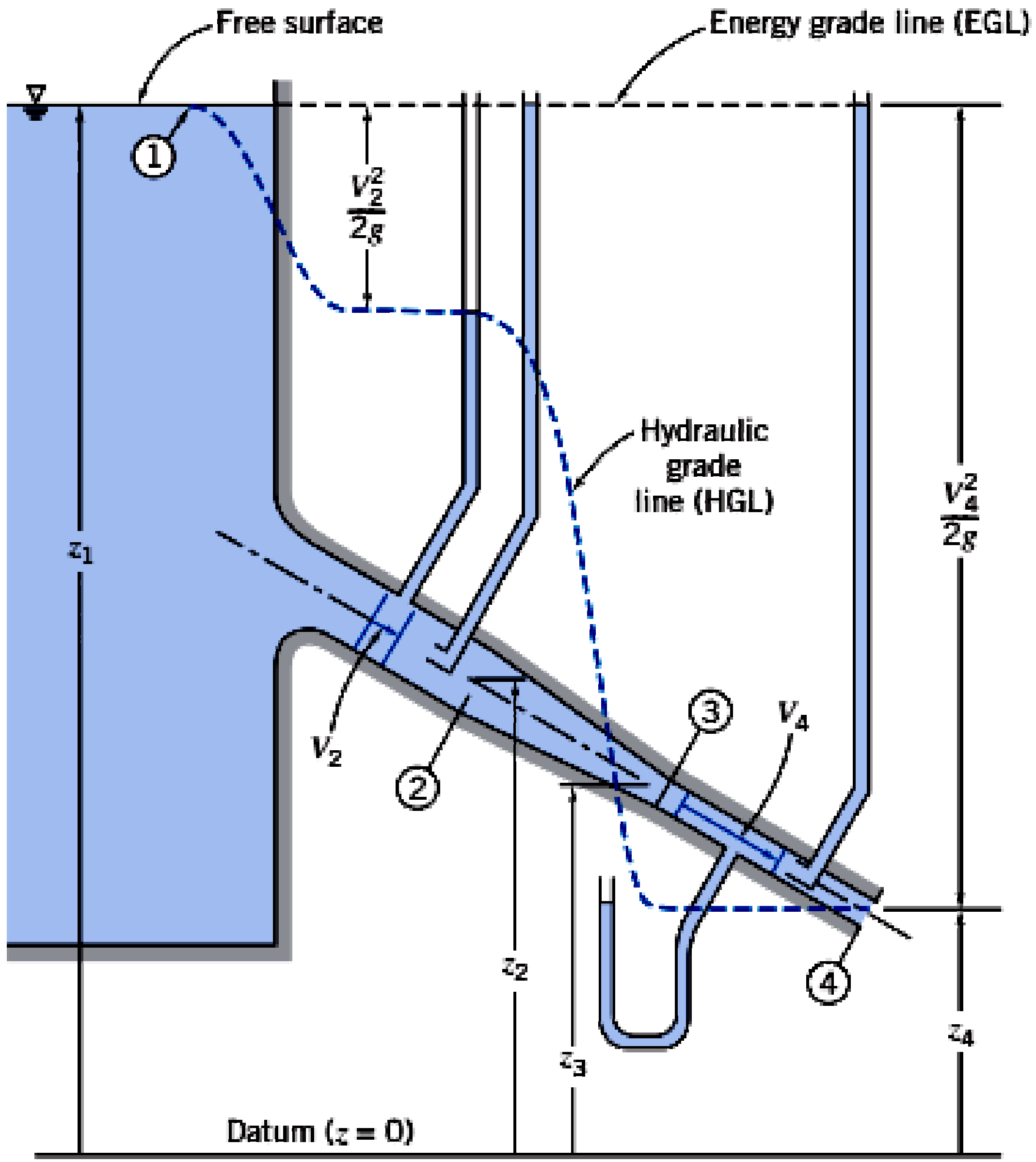


Fig: Energy and hydraulic grade lines for **frictionless flow**.

The fluid now has significant kinetic energy and has lost some potential energy (can you determine from the figure what happened to the pressure?). From Eq. (5c), we can see that the HGL is lower than the EGL by $V_2^2 / 2g$; the HGL at station 2 shows this.

- From station 2 to station **3 there is a reduction in diameter**, so continuity requires that $\mathbf{V_3 > V_2}$; hence **the gap between the EGL and HGL increases** further, as shown.

- Station 4 is **at the exit (to the atmosphere)**. **Here the pressure is zero (gage), so the EGL consists entirely of kinetic and potential energy terms**, and $HGL_4 = HGL_3$. ***We can summarize two important ideas when sketching EGL and HGL curves:***

1. The EGL is constant for incompressible, inviscid flow (in the absence of work devices). We will see further that ***work devices may increase or decrease the EGL, and friction will always lead to a fall in the EGL***.

2. The HGL is always lower than the EGL by distance $V^2/2g$. Note that the value of velocity ***V*** **depends on the overall system** (e.g., **reservoir height, pipe diameter, etc.), but changes in velocity only occur when the diameter changes**.

Example-2:

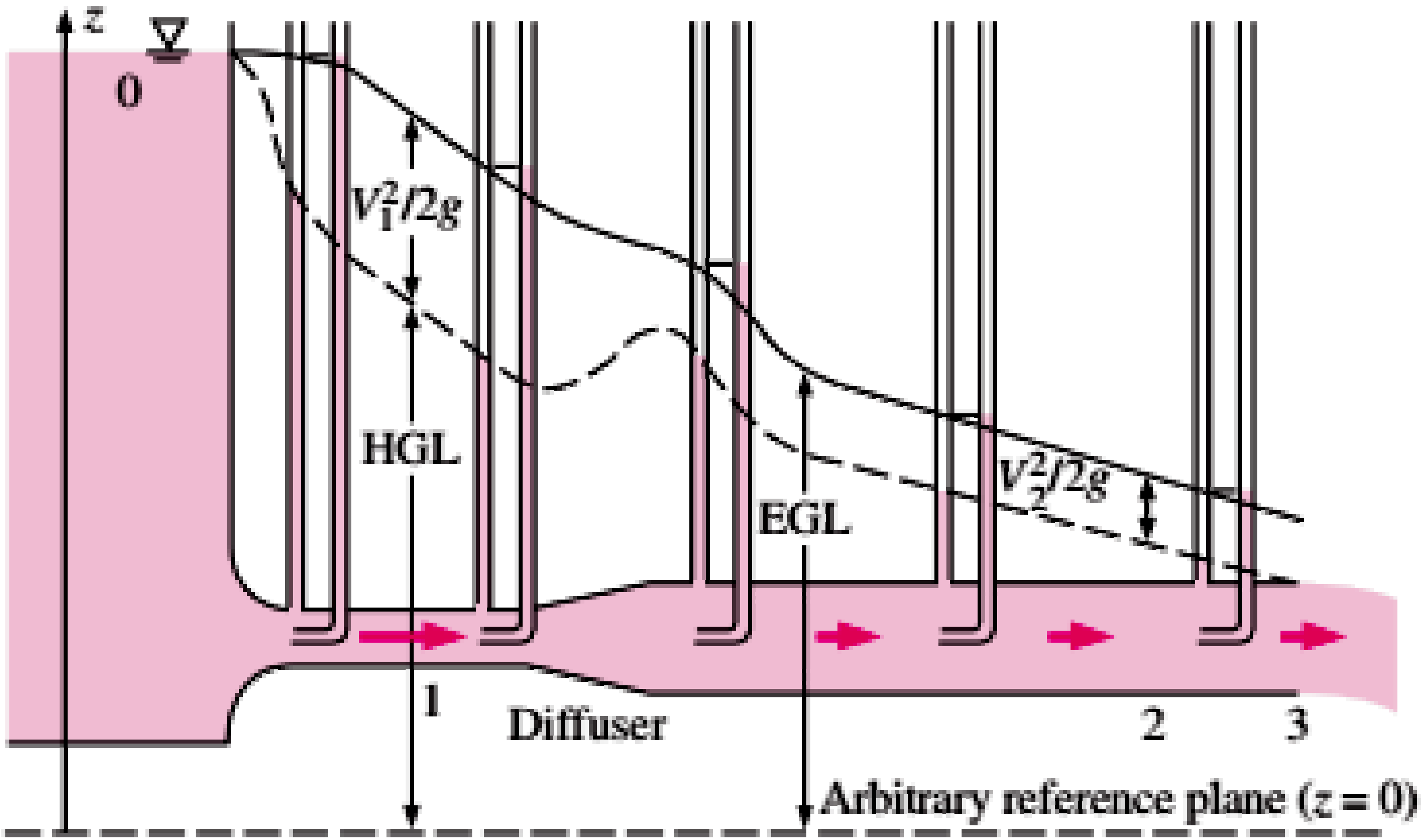


Fig: 1 (real flow)

At point 0 (at the liquid surface), EGL and HGL are even with the liquid surface since there is no flow there. HGL decreases rapidly as the liquid accelerates into the pipe; however, EGL decreases very slowly through the well-rounded pipe inlet. ***EGL declines continually along the flow direction due to friction and other irreversible losses in the flow. EGL***

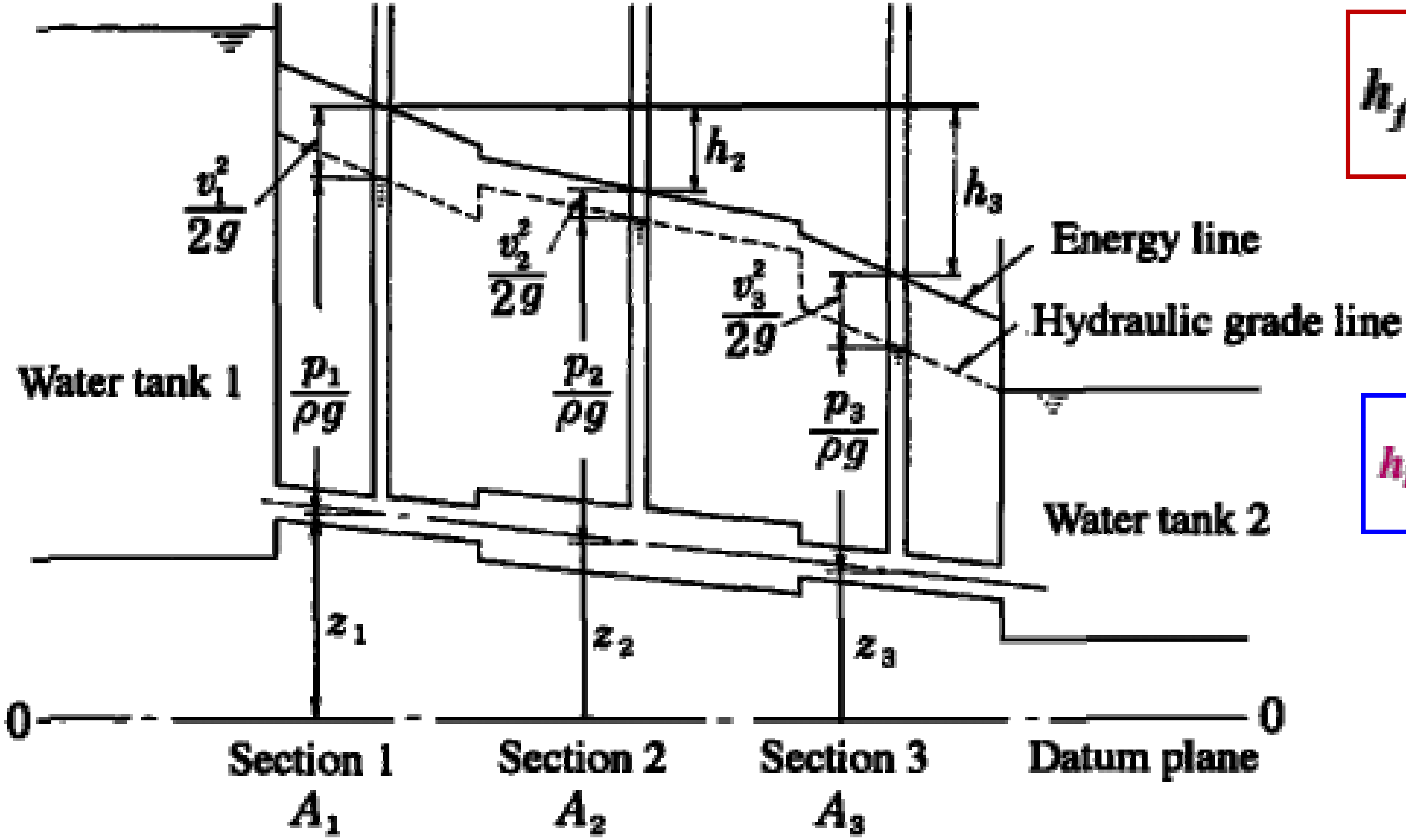
cannot increase in the flow direction unless energy is supplied to the fluid. HGL can rise or fall in the flow direction, but can never exceed EGL. HGL rises in the diffuser section as the velocity decreases, and the static pressure recovers somewhat; the total pressure does *not* recover, however, and EGL decreases through the diffuser. The difference between EGL and HGL is $V^2_1/2g$ at point 1, and $V^2_2/2g$ at point 2. Since $V_1 > V_2$, the difference between the two grade lines is larger at point 1 than at point 2. The downward slope of both grade lines is larger for the smaller diameter section of pipe since the frictional head loss is greater. Finally, HGL decays to the liquid surface at the outlet since the pressure there is atmospheric. However, EGL is still higher than HGL by the amount $V^2_2/2g$ since $V_3 = V_2$ at the outlet.

Example-3:

As shown in following Fig., whenever water flows from tank 1 to tank *2,* the energy equations for sections 1 , 2 and 3 are as follows:

$$\boxed{\frac{v_1^2}{2}+\frac{p_1}{\rho}+z_1=\frac{v_2^2}{2}+\frac{p_2}{\rho}+z_2+h_2=\frac{v_3^2}{2}+\frac{p_3}{\rho}+z_3+h_3}$$

$$h_f = \frac{fLv^2}{2gd}$$



$$h_L = K_L \frac{V^2}{2g}$$

Fig: Hydraulic grade line and energy line (real)

- Where h_2 and h_3 are the losses of head between section 1 and either of the respective sections. In Fig., the line connecting the height of the pressure heads at respective points of the pipe line is called the hydraulic grade line, while that connecting the heights of all the heads is called the energy line.

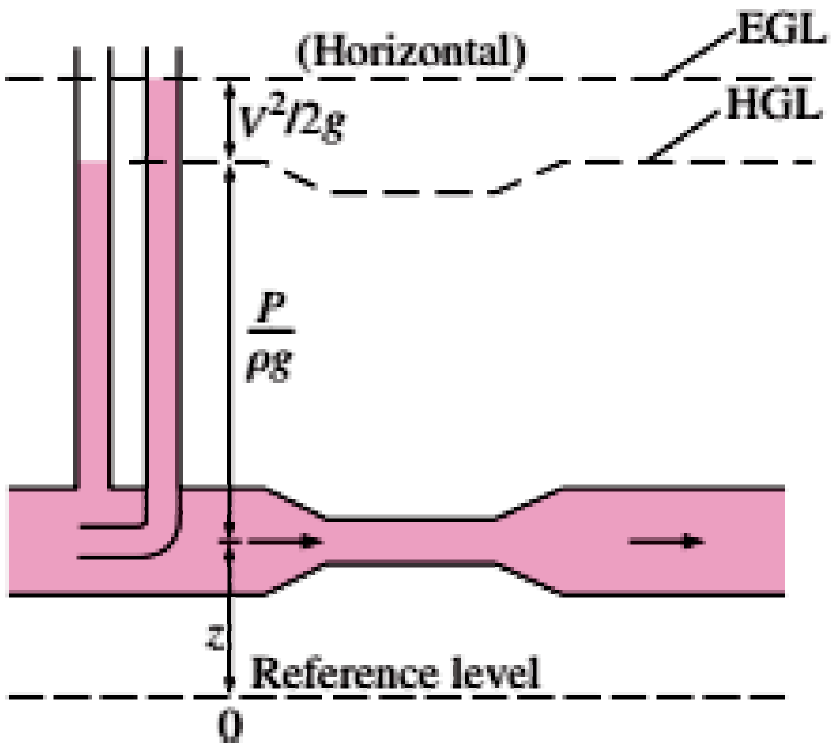
We note the following about the HGL and EGL:

- For ***stationary bodies such as reservoirs or lakes, the EGL and HGL coincide with the free surface of the liquid***. The elevation of the free surface *z* in such cases represents both the EGL and the HGL **since the velocity is zero and the static pressure (gage) is zero.**

- The EGL is always a distance $V^2/2g$ above the HGL. These two lines approach each other as the velocity decreases, and they diverge as the velocity increases. The height of the HGL decreases as the velocity increases, and vice versa.

- In an idealized Bernoulli-type flow, EGL is horizontal and its height remains constant. This would also be the case for HGL when the flow velocity is constant.

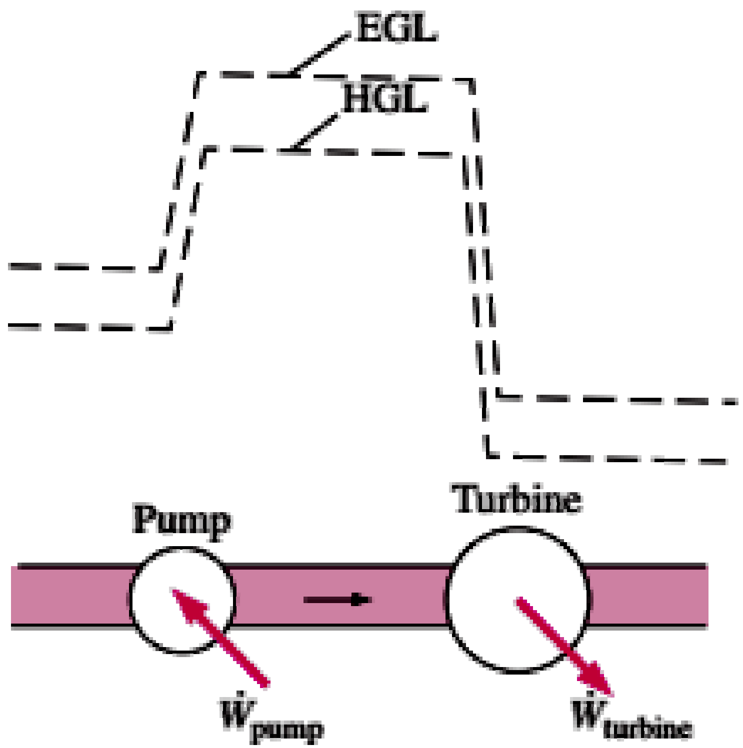
- **For open-channel flow**, the HGL coincides with the free surface of the liquid, and the EGL is a distance $V^2/2g$ above the free surface.



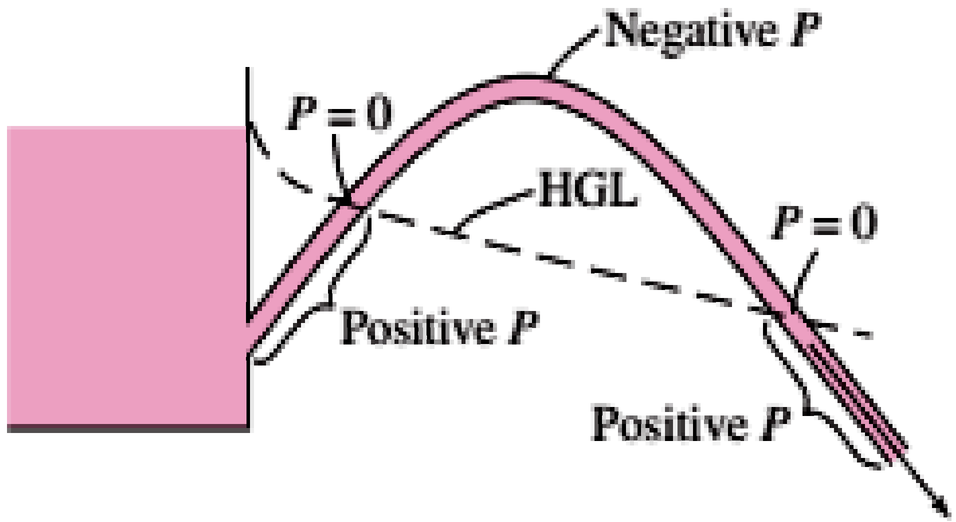
- At a *pipe exit,* the pressure head is zero (atmospheric pressure) and thus the HGL coincides with the pipe outlet (location 3 on Fig. 1).

- **The mechanical energy loss due to frictional effects (conversion to thermal energy) causes the EGL and HGL to slope downward in the direction of flow.** The slope is a measure of the head loss in the pipe. A component that generates significant frictional effects such as a valve causes a sudden drop in both EGL and HGL at that location.

- ***A steep jump occurs in EGL and HGL whenever mechanical energy is added to the fluid (by a pump, for example***). Likewise, ***a steep drop occurs in EGL and HGL whenever mechanical energy is removed from the fluid (by a turbine***, for example), as shown in Fig.



- The pressure (gage) of a fluid is zero at locations where the HGL intersects the fluid. The pressure in a flow section that lies above the HGL is negative, and the pressure in a section that lies below the HGL is positive (Fig.). Therefore, an accurate drawing of a piping system and the HGL can be used to determine the regions where the pressure in the pipe is negative (below the atmospheric pressure).



Limitations on the Use of the Bernoulli Equation

The Bernoulli equation is one of the most frequently used and misused equations in fluid mechanics. Its versatility, simplicity, and ease of use make it a very valuable tool for use in analysis, but the same attributes also make it very tempting to misuse. Therefore, it is important to understand the restrictions on its applicability and observe the limitations on its use, as explained here:

1. Steady flow: The first limitation on the Bernoulli equation is that it is applicable to *steady flow*. Therefore, it ***should not be used*** during ***the transient start-up and shut-down periods, or during periods of change***

in the flow conditions.

2. <u>Frictionless flow</u>: Every flow involves some friction, no matter how small, and *frictional effects* may or may not be negligible. In general, frictional effects are negligible for short flow sections with large cross sections, especially at low flow velocities. ***<u>Frictional effects are usually significant in long and narrow flow passages</u>***, ***<u>in the wake region downstream of an object</u>***, and ***<u>in diverging flow sections such as diffusers</u>*** because of the increased possibility of the fluid separating from the walls in such geometries. Frictional effects are also significant ***<u>near solid surfaces</u>***, and thus **<u>the Bernoulli equation is usually applicable along a streamline in the core region of the flow, but not along a streamline close to the surface</u>** (Fig. 2).

A ***<u>component that disturbs the streamlined structure of flow</u>*** and thus ***<u>causes considerable mixing and backflow such as a sharp entrance of a tube or a partially closed valve</u>*** in a flow section can make the Bernoulli equation inapplicable.

3. <u>No shaft work</u>: The Bernoulli equation was derived from a force balance on a particle moving along a streamline. Therefore, the Bernoulli

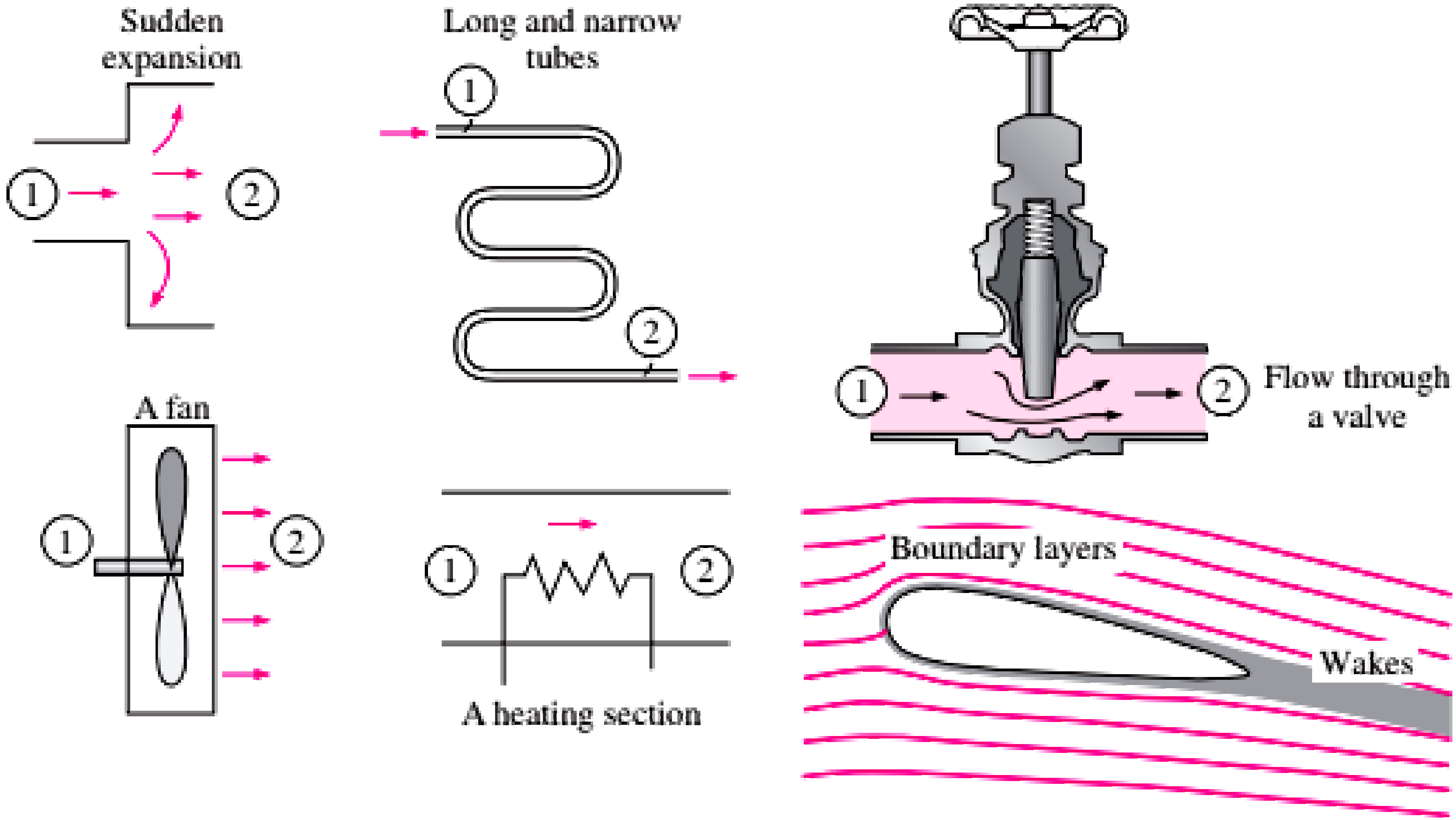


Fig.2: Flow section make the Bernoulli equation invalid.

equation ***is not applicable in a flow section that involves a pump, turbine, fan, or any other machine or impeller since such devices destroy the streamlines*** and carry out energy interactions with the fluid particles.

However, the Bernoulli equation can still be applied to a flow section prior to or past a machine (assuming, of course, that the other restrictions on its use are satisfied). In such cases, the Bernoulli constant changes from upstream to downstream of the device.

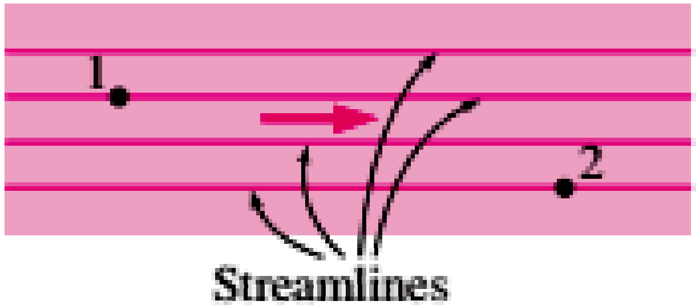
4. Incompressible flow: The assumptions that ρ = constant i.e. the flow is incompressible is satisfied by liquids and also by gases at Mach numbers less than about 0.3 since compressibility effects and thus density variations of gases are negligible at such relatively low velocities. Note that there is a compressible form of the Bernoulli equation.

***Unsteady, compressible flow*:**

$$\boxed{\int \frac{dP}{\rho} + \int \frac{\partial V}{\partial t} ds + \frac{V^2}{2} + gz = \text{constant}}$$

5. No heat transfer: The density of a gas is inversely proportional to temperature, and thus the Bernoulli equation should not be used for flow sections that involve significant temperature change such as heating or cooling sections.

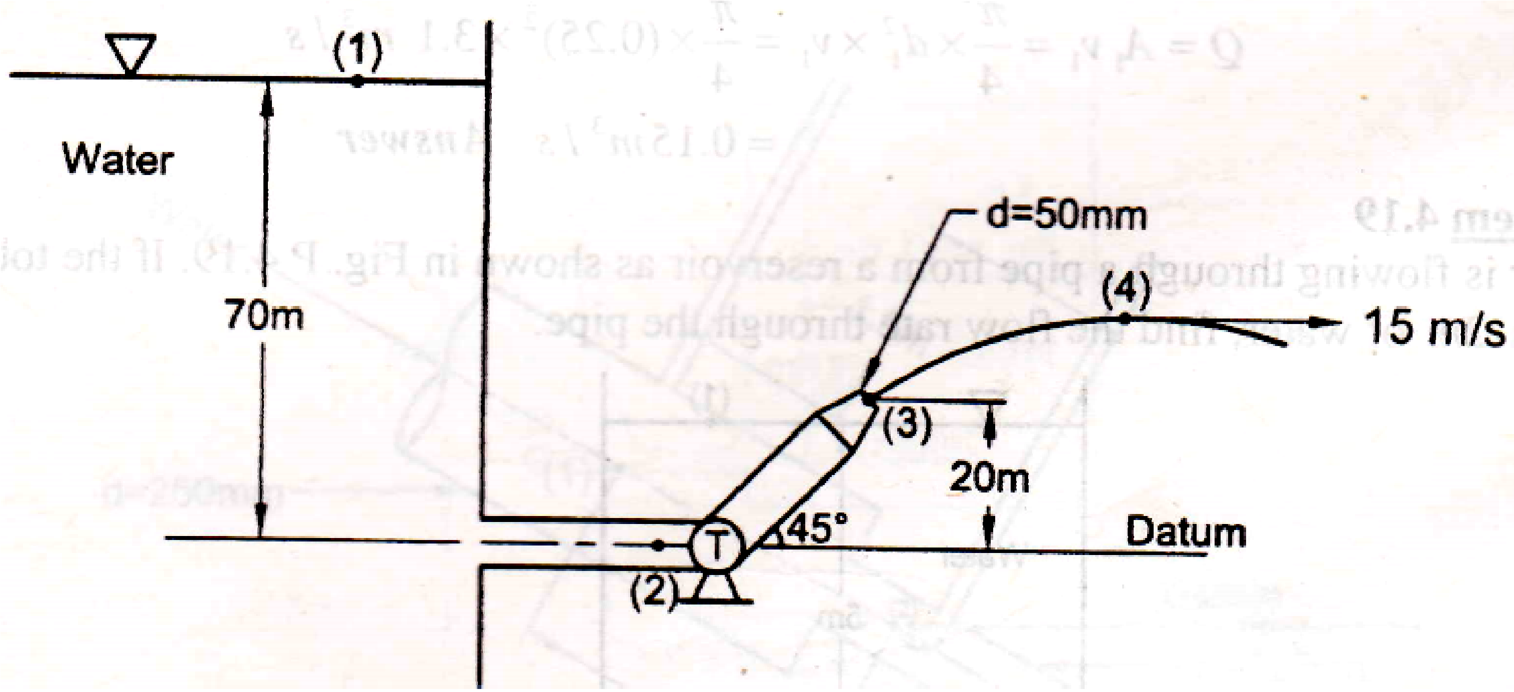
6. <u>Flow along a streamline</u>: Strictly speaking, the Bernoulli equation ***$P/\rho + V^2/2 + gz = C$*** ***<u>is applicable along a streamline</u>***, and ***<u>the value of the constant C, in general, is different for different streamlines.</u>*** But when a region of the **<u>flow is *irrotational*, and thus there is no *vorticity*</u>** in the flow field, **<u>the value of the constant *C* remains the same for all streamlines</u>**, and, therefore, the Bernoulli equation becomes applicable *across* streamlines as well.



$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Problem:

Water is flowing through a turbine as shown in Fig. Find the power developed by the turbine. Neglect all losses.



Given Data:

Height of water level, H = 70 m

Diamerer of pipe at section 2, d_2 = 100mm

Diamerer of pipe at section 3, d_3 = 50mm

Horizontal velocity of water at point 4, V_4 = 15 m/s

Solution:

Relation between velocity of water at points 3 and 4,

$$V_4 = V_3 \cos 45^0$$
$$\text{or, } 15 = V_3 \times 0.707$$
$$\text{or, } V_3 = 21.22\, m/s$$

Flow rate of water,

$$Q = A_3 V_3 = \frac{\pi}{4} \times d_3^2 \times V_3 = \frac{\pi}{4} \times (0.05)^2 \times 21.22\, m^3/s = 0.042\ m^3/s$$

Applying Bemoulli's equation between points 1 and 2, we have

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

$$\text{or, } 0 + 0 + 70 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + 0$$

$$\text{or, } 70 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} \qquad \text{----------} \quad \textbf{(i)}$$

Again applying Bernoulli's equation between points 2 and 3,

$$\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+Z_2=\frac{p_3}{\gamma}+\frac{V_3^2}{2g}+Z_3+E_T$$

$$\text{or, } \frac{p_2}{\gamma}+\frac{V_2^2}{2g}+0=0+\frac{V_3^2}{2g}+20+E_T$$

$$\text{or, } \frac{p_2}{\gamma}+\frac{V_2^2}{2g}=\frac{V_3^2}{2g}+20+E_T \quad \text{----------} \quad \textbf{(ii)}$$

From equations (i) and (ii),

$$70=\frac{V_3^2}{2g}+20+E_T$$

$$\text{or, } 50=\frac{(21.22)^2}{2\times 9.81}+E_T$$

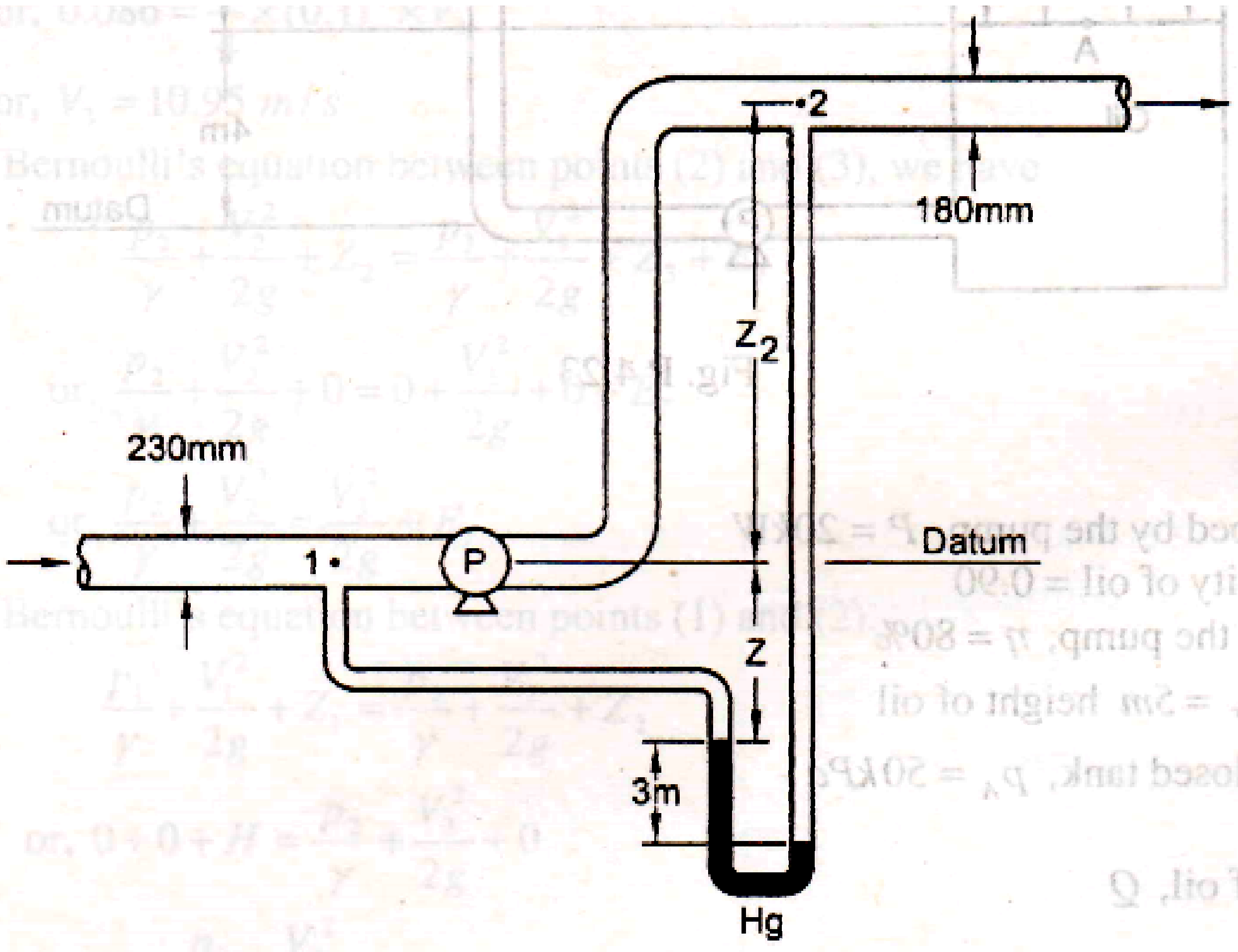
$$\text{or, } E_T=27.05\ \frac{kN\,m}{kN}$$

Power developed by the turbine, P = γQH

$$P = Q\gamma E_T$$
$$= 0.042 \times 9.81 \times 27.05\,kW$$
$$= 11.15\,kW \quad \textbf{\textit{Answer}}$$

Problem:

Water is flowing at the rate of 0.15 m^3/s through the piping system as shown in Fig. Find the power required by the pump.



Given Data:

Diameter of inlet pipe, $d_1 = 230$ mm
Diameter of outlet pipe, $d_2 = 180$ mm
Deflection in manometer, h = 3m
Flow rate of water, Q = 0.I5 m^3/s

Solution :

Let γ = specific weight of water = 9.8l kN/m^3
From manometry, we have

$$p_2 + Z_2\gamma + Z\gamma + 3\gamma = p_1 + Z\gamma + 3\times 13.6\times\gamma$$

$$\text{or, } p_2 + Z_2\gamma - 3\times 12.6\times\gamma = p_1$$

$$\text{or, } \frac{p_2 - p_1}{\gamma} = 3\times 12.6 - Z_2 \qquad \text{------- (i)}$$

Applying Bernoulli,s equation between sections I and 2, we have

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 + E_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

$$\text{or, } \frac{p_2 - p_1}{\gamma} + \frac{V_2^2}{2g} + Z_2 = \frac{V_1^2}{2g} + E_p \qquad \text{---------- (ii)}$$

Here $Z_1 = 0$

From equations (i) and (ii), we have

$$3\times 12.6 - Z_2 + \frac{V_2^2}{2g} + Z_2 = \frac{V_1^2}{2g} + E_p$$

$$\text{or, } 3\times 12.6 + \frac{V_2^2}{2g} = \frac{V_1^2}{2g} + E_p \quad \text{----------} \quad \textbf{(iii)}$$

From continuity equation for incompressible fluid, we have,

$$A_1 V_1 = A_2 V_2$$

$$\text{or, } \frac{\pi}{4}\times d_1^2 \times V_1 = \frac{\pi}{4}\times d_2^2 \times V_2$$

$$\text{or, } \frac{\pi}{4}\times (0.23)^2 \times V_1 = \frac{\pi}{4}\times (0.18)^2 \times V_2$$

$$\text{or, } V_2 = 1.63 V_1$$

Again flow rate $Q = A_1 V_1$

$$\text{or, } V_1 = \frac{0.15}{\frac{\pi}{4}\times (0.23)^2} = 3.61 m/s$$

Considering the equation (iii), we have

$$3\times12.6+\frac{(1.63V_1)^2}{2g}=\frac{V_1^2}{2g}+E_p$$

$$\text{or, } 3\times12.6+\frac{(1.63)^2(3.61)^2}{2\times9.81}=\frac{(3.61)^2}{2\times9.81}+E_p$$

Or, $E_p = 38.9\ kNm/kN$

Power required by pump, $P = Q\,\gamma\,E_p$

$= 0.15 \times 9.81 \times 38.9$ kW

$= 57.24$ kW. Answer